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TENSOR NETWORK THEORY OF THE
METAORGANIZATION OF FUNCTIONAL GEOMETRIES
IN THE CENTRAL NERVOUS SYSTEM

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Abstract—Here we present an elaboration and a quantitative example for a hypothetical neuronal process, implementing what we refer to as the metaorganization principle. This process allows the internalization of external (body) geometries into the central nervous system (CNS) and a reciprocal and equally important action of the CNS geometry on the external (body) geometry. The hypothesis is based on the distinction, within the CNS, between covariant sensory and contravariant motor vectorial expressions of the extrinsic geometry. These sensory and motor expressions, given in natural co-ordinate systems, are transformed from one to the other by a neuronal network which acts as a metric tensor. The metric tensor determines the relationship of these two expressions and thus comprises the functional geometry of the system.

The emergence through metaorganization of networks that implement such metric function is viewed as the result of interactions between the covariant motor execution which generates a physical action on the external world (via the musculoskeletal system) and the covariant sensory proprioception which measures the effect of such motor output. In this transformation of contravariants to covariants by the physical geometry of the motor system, a covariant metric tensor is expressed implicitly. However, co-ordinated motor action requires its dual tensor (the contravariant metric) which is assembled in the CNS based on the metaorganization principle, i.e. the ability of CNS and external geometries to mold one another. The two metric transformations acting on each other detect error signals whenever the match of the physical and functional geometries is imperfect. Such error signals are utilized by the metamodularization process to improve the match between the two metrics, so that with use the internal representation becomes increasingly homeometric with the geometry of the external world.

The proposed physical process by which the metaorganization principle is implemented is based on oscillatory reverberation. If covariant proprioception is used as a recurrent signal to the motor apparatus, as if it were a contravariant motor expression, then reverberations at their steady-state yield the eigenvectors and eigenvalues of the system. The stored eigenvectors and eigenvalues can serve, respectively, (1) as a means for the genesis of a metric (in the form of its spectral representation) with the given eigenvectors and (2) as a means of comparing the eigenvalues that are implicit in the external body geometry and those of the internal metric. The difference between these eigenvalues is then used to modify the metric so that it "evolves" to perform a more accurate covariant-contravariant transformation.

The metric can be represented by the dyadic outer products of its eigenvectors where each dyad is weighted by the corresponding eigenvalue. Such a spectral representation yields in a uniform manner (a) the contravariant metric (in the case of complete CNS hyperspaces) and (b) the Moore-Penrose generalized inverse of the covariant metric tensor (in the case of non-Riemannian overcomplete CNS hyperspaces).

The metaorganization of metric networks, together with the sensorimotor covariant embedding network, provides an explanation of the emergence of a whole sensorimotor apparatus leading to the feasibility of constructing truly brain-like robotic systems.

1. INTRODUCTION

1.1 Geometries and brain function

The evolvement of the CNS through natural selection is the fundamental means by which multicellular organisms develop optimal interactions with the world. In terms of abstract geometry this can be expressed by stating that the function of the brain is to match the system of relations among objects in the external world, with a multidimensional inner functional geometry, in a manner such that these geometries approach homeomorphism if not isomorphism.25,80 While general considerations as such may be of significance in brain theory, the nature of the interaction between the inner and outer geometries must be defined in a concrete manner.

Sensorimotor operations are the appropriate paradigms of brain function to consider first for such elaboration. Here, the relation between the external world and its internal representation in the brain can be directly observed and quantitatively treated as it is expressed by the precision of goal-oriented movements.

Sensorimotor transformations, most particularly those involving the cerebellum, have already been investigated from the point of view of how they adapt to an alteration of geometries, either of the external physical arrangement or its inner representation. As an example of the former, modification of the vestibulo-ocular reflex, as in the case of vision inverted by reversing prisms, has been amply studied.23 For the latter the compensation for unilateral
vestibular ablation has also been analysed in detail. The issue in this paper is to define the mechanism by which the relationship between inner brain geometry and the external world are matched in accordance with the modified conditions. This point has been raised in preliminary communications.75-82,85

1.2. The essentials for a geometrical approach to brain function

The general hypothesis of the geometrical interpretation of brain function hinges on the assumption that the relation between the brain and the external world is determined by the ability of the CNS to construct an internal model of the external world using an interactive relationship between sensory and motor expressions. This relation is evident, for instance, in the orienting-response of an animal in a new environment. Indeed the process of sensory detection involves a significant amount of motor activity.32 Such observations strongly indicate that the sensorimotor transformations are the gauging tool by which the CNS relates to the external world. It has been demonstrated in detail in human and animal experimentation that a convergence of sensory and motor information is necessary to form internal models of novel objects.23,63

At the level of mathematical analysis, the basis for formalizing the above in terms of a geometrical representation appears to be the dual expression of extrinsic physical objects by intrinsic "CNS vectors". These vectorial expressions, with respect to reference frames intrinsic to the organism, are covariant for sensory analysis and contravariant for motor synthesis. The geometrical relationship between these two vectorial components is comprised in a neuronal network acting as a metric tensor.51

1.2.1. Fundamental mathematical concepts inherent in covariant and contravariant neuronal representations. The covariant-contravariant distinction is of cardinal importance, as their relationship determines the metric, which comprises the functional geometry of the system. Indeed these two ways in which the CNS relates to the external world are profoundly different. The primary expression derives from an external object a multitude of covariant vector components, which constitutes a sensory input to the CNS. Such a vector arises from the firing frequencies of a set of sensory neurons which collectively represent an extrinsic physical state. The secondary expression is the implementation of a physical reality from a multitude of contravariant components; e.g. generating a displacement of the arm through the activation of individual motor units. This dual relationship between CNS expressions and physical invariants via the sensory and motor systems is fundamental. The question is not whether such different expressions exist but, rather, whether it is possible to construct a precisely formulated general concept which encompasses the functional essence of (a) the above sensory and motor properties and (b) the transformations which convert one into the other. The most fundamental difference between these two expressions appears to be their causality; i.e. that in the former the components arise from the physical reality, while in the latter the physical reality arises from the components.

Sensory reception is therefore an analysis (i.e. a differentiation) while motor execution is a synthesis (i.e. an integration). The covariant expression is based on a process of comparing the features of the external physical reality (which is co-ordinate-system-invariant) with a set of internal physical states (which serve as the measuring standard within the CNS). Each of these relationships yields a quantity which is then used as an inner measure of one aspect of the external reality. An example of this procedure in the CNS is the inertial force generated by the endolymphatic mass in each semicircular canal, each yielding one individual measure (a cosine component) of the head-acceleration. Note that as defined in calculus, establishing the relation of the rates of change is a differentiation. Given two invariants their infinitesimal relationship yields the derivative. It is also well known that, following the chain-rule of differentiation, such derivative components of x change with the alteration of the frame a in a manner obeying the following rule of covariant transformation:14,94,101

\[ x' = (\partial a'/\partial a) \times x. \] (1)

The biological significance of these known mathematical properties was expressed in the proposal that these projection-type covariant components correspond to sensory processes.39

Motor actions, on the other hand, are integrative-type operations. In a motor process such as displacement of the eye by the co-contraction of the extraocular muscles, the physical summation of the components is of the essence and not the differential-type sensory relationship of one physical invariant to another. Infinitesimally, the summation is implemented by the parallelogram components that obey the rule of covariant transformation.14,101

\[ x' = (\partial a'/\partial a) \times x'. \] (2)

Reference frames are invoked by both the sensory and motor processes. These are instruments, intrinsic to the structure of the organism, through which extrinsic points of physical spaces are related to arithmetical manifolds within the CNS.

Note that so far the only requirement in the interpretation of sensory covariants and motor contravariants is that the derivatives of the manifold should exist, i.e. that the manifold which arises from the use of reference frames be smooth. This is the necessary and sufficient prerequisite for the existence of covariant sensory and contravariant motor expressions.

Transformation of co- and contravariants through the metric is a different problem from the simple
distinction between these two types of representation. The central question is whether a concise expression can be given for the transformation between these co- and contravariant vectorial forms. As is well known, such expressions can be obtained since any geometry can be most concisely characterized by its metric tensor (also called fundamental tensor), which expresses the relationship between covariants and contravariants. Indeed, a study of any geometry may well start with the definition of its metric. It has been stressed, however, that the geometry of the CNS hyperspaces cannot be a priori characterized by a known metric; e.g. by an Euclidean metric (as in Cartesian tensor analysis that is used in engineering) or by a Riemannian metric (as in the tensor analysis of four-dimensional manifolds used in relativity theory). The features of the metric in a mathematical CNS hyperspace cannot be taken for granted; in fact the metric is the unknown in brain research. Indeed if a CNS hyperspace is amorphous, a metric in the strict sense may not even exist; sensory covariants and motor contravariants may be unrelated (which would manifest itself in a dysmetric motor action). The existence of any metric in the CNS poses the question to the experimenter "how to find it" and to the theorist, "how such metric is developed in complex organisms".

1.2.2. Fundamental differences in biological interpretation of the contravariant motor activation and covariant proprioception, commensurate with tensions. The metaorganization principle will be elaborated in this paper by means of a quantitative illustration. The model diagram shown in Fig. 1 has been used in a preliminary form in preceding publications on tensor network theory. First it will be shown, by means of a specific quantitative example, that the active forces that are exerted by the muscles upon inner-vation are proportional to the contravariant physical components of the impressed load G. In contrast, the passive forces that are measurable by proprioceptive peripheral sensory system, can be determined proportionally to covariant orthogonal projection components of G.

The motor execution mechanism, shown schematically in Fig. 1(A), assumes that the three-joint arm moves in the two-dimensional plane of the paper, meaning that the motor system is overcomplete. A flexor and an extensor muscle at each of the three joints are depicted in Fig. 1(A): the shoulder, elbow and wrist. The pectoralis muscle (1) acts as a flexor, and the deltoideus (2) acts as an extensor of the upper arm at the shoulder joint. Thus the difference between the forces exerted by them would be the force acting at the index finger in direction alpha (Fig. 1B). Likewise, the triceps muscle (3) extends while the brachialis (4) flexes the lower arm and thus the resultant force is along direction beta. In turn the extensor carpi ulnaris (5) generates a force along direction gamma, working against the force of the flexor carpi radialis (6).

The x, y and z directions shown in Fig. 1(B) determine a local non-orthogonal frame of reference. The general considerations in 1.2.1. are demonstrated in detail, in Figs 1(C) and (D), namely in such non-orthogonal frames of reference two vectorial expressions are possible, each with different characteristics. If a load force (G) is attached to the index finger (Fig. 1C), and it is assumed that the arm holds this load in a stationary position, both the active and passive forces are expressed in the (position-dependent) local system of co-ordinates as introduced in Fig. 1(B). However, the active motor actions rotate the arm-segments by means of the forces exerted by the muscles (Fig. 1C), whereas peripheral proprioceptive organs detect different, passive forces (Fig. 1D).

The contravariant character of the forces exerted by the muscles follows directly from the fact that in a steady-state the active forces must balance; the muscle components must yield a resultant force that is equal but opposite in direction to the load (G, see Fig. 1C). These physical components of the load (G) are, by definition, the contravariant components that add according to the parallelogram rule. The forces of the muscles have always been regarded to be the physical components that balance the load (G). Tensor theory simply re-phrases this by stating that the vectorial expression of an object in any system of co-ordinates by means of physical components is of the contravariant type. It has been discussed previously, and is obvious from Fig. 1(C), that such a contravariant expression of a given load (G) in an overcomplete frame of reference is not unique. Indeed, an infinite number of configurations of the components along x, y and z can result in the same load (G). In fact it is this mathematical indeterminacy that raises the question of how does the CNS arrive, in a co-ordinated act, at a particular choice from an infinite number of possible solutions. The most obvious demonstration of such overcompleteness can be shown by remembering that any constant can be added to the forces exerted by both an agonist (flexor) and antagonist (extensor) muscle-pair, without changing the resultant, since the two forces act along a common co-ordinate axis and thus the additions cancel one another. While such addition to both components of a pair of muscles is mathematically redundant, its physiological significance is obvious. The same stationary position of the arm can be held with the muscles contracting minimally, or in an overexerted stiff manner; consequently, the holding position may be delicate or robust. The understanding that additions of cancelling force-pairs will not modify position makes it possible to consider in the rest of the paper only the differences of forces in reciprocally innervated muscle pairs acting along common axes. Such "pairing", however, is only for simplifying convenience; the scheme to be presented in this paper also applies to the separate treatment of muscles acting along different axes.
Physical Frames of Reference and Types of Coordinates Intrinsic to Motor Apparatus

Frames of Reference

A. Pellionisz and R. Linás

Physical Frames of Reference and Types of Coordinates Intrinsic to Motor Apparatus

Fig. 1. Schematic representation of the physical geometry of an exemplary motor executor apparatus, which is to be matched by a functional geometry (implemented by the cerebellum), their match enabling the co-ordinated control of the multiarticulate limb. (A) and (B) show the frame of reference of limb-displacements, that is intrinsic to the motor apparatus. (C)-(E) demonstrate the two different kinds of co-ordinates (contravariant and covariant) that express the physical invariant of a displacement by active muscle components and passive proprioceptive components, respectively—the two vectorial expressions implying the covariant metric inherent in the physical geometry of the motor apparatus. (A) Individual muscles determine a local curvilinear frame of reference for displacements of the limb by six major muscles. 1, pectoralis; 2, deltoideus; 3, triceps; 4, brachialis; 5, extensor carpi ulnaris; 6, flexor carpi radialis. (B) Simplified, non-orthogonal rectilinear frame of reference of limb-movements. The α, β and γ local displacement-directions belong to individual incrementation of the joint-angles. (C) Physical (contravariant, parallelogram-type) vectorial expression of an invariant $G$ in the intrinsic frame of reference. (D) Proprioceptive (covariant, projection-type) expression of the same $G$; with unique components even in an overcomplete frame. (E) Contravariant components physically execute motor acts, covariant components are in turn detectable by proprioception. Thus a contravariant metric tensor is implied in the physical geometry of a motor apparatus.

It must be emphasized that in the non-orthogonal system of co-ordinates shown in Fig. 1, in addition to the contravariant active physical muscle forces, a different covariant-type vectorial expression of the load ($G$) is also possible (Fig. 1D). The covariants are the passive force-components, measured as the orthogonal projections of the load-vector ($G$) onto the co-ordinate axes. The nature and the functional role of the passive forces, detectable by peripheral proprioceptive systems such as the tendon organs, have not been conclusively defined in motor physiology, despite thorough analysis.3,8,9 The prevalent ideas are that this system monitors the load on the motor apparatus or provides an overload-preventing warning signal. The role of proprioception is therefore particularly intriguing in oculomotor actions, where the load of the system is constant.3,8 In contrast, it has been suggested in tensor theory of the CNS that proprioception serves to supply the components of the dual complementsor motor vector, the covariant counterpart of the contravariant motor action.3

It is well known that peripheral receptor organs are capable of measuring passive stretch. Such a passive force may significantly differ from the active force exerted by the given muscle. For instance, while the active force generated by a muscle is proportional to its own motoneuron-activation, a stretch in the tendon of the given muscle arises from an interaction of the activity of many muscles as well as from external load factors. Since the tendons utilize the same $x$, $β$ and $γ$ local system of co-ordinates as the motor actuators, and they express the physical object...
of the load in a sensory manner, they yield the
covariant components of the load. This proposal is
substantiated below with the help of Fig. 1(E).
Muscles exert force on the skeletal system (even if
the action arises from a variety of active and passive
factors) through their tendons. Thus, if a load \( G \) is
balanced in a stationary position, the passive force
\( F \), proportionally detectable at the elbow-joint
denoted by \( \beta \), would be the difference of the tendon-
forces of the extensor and flexor carpi. This passive
force \( F \), measurable by a strain-gauge in the tendon
at \( \beta \), must balance the torque \( (Gg) \), exerted by the
load, by \( (Fj) \) (cf. Fig. 1E). From the similarity of the
GF and Gf triangles, it follows that \( F = G \cos (\omega) \),
which is the definition of the covariant, orthogonal
projection-type component. Note that the set of
covariant proprioception components of \( G \), measured
as the tensions in the tendons, is unique, in contrast
to the non-uniqueness of the contravariant com-
ponents.
The scheme in Fig. 1 presents a motor effector
mechanism in which the dual sets of co- and contra-
variant components are available. This poses the
following challenge to the CNS: given a contra-
variant motor execution vector the proprioceptive
system must provide the corresponding covariant
vector. Such contravariant-to-covariant relation is
implied in the physical geometry of the effector
mechanism. Thus when an internal functional geom-
etry is developed by the CNS the physical geometry
must be matched by this homeometric internal
representation. It is suggested that the matching of
the physical geometry with its functional counterpart is
furnished by the cerebellum. Below, a concise
account is given of how the cerebellar circuit may
perform as a co-ordinator once the circuit is avail-
able (point 1.2.3) and how such explanation leads
to the question of how such networks may emerge
(point 1.3).

1.2.3. Utilization of dual vectorial representations to
explain CNS functions: the tensor model of cerebellum.
Tensor network theory of the CNS evolved as a
mathematical formulation, with the use of the above
basic terms of covariants and contravariants, of the
geometrical concept of brain function, especially that
of the cerebellum.\(^{74-78, 80-83}\) The tensor model can
conscisely explain in the above terms the function of
the cerebellar circuit, once that circuit is available
through the development of a sensorimotor system.
A general tensorial interpretation of the CNS is
based on the notion that the intrinsic natural frames of
reference, in which neurons attribute ordered sets of
activity-values (co- and contravariant vectors) to
physical invariants of the external world, invoke
multidimensional arithmetic manifolds. The func-
tional geometry of such a CNS hyperspace is com-
prised by its metric tensor, which can be implemented
by a matrix—perhaps the most natural abstract
representation of a neuronal network. Sensorimotor
systems could therefore be functionally explained in
a three-stage scheme.\(^{81}\) This consists of (1) a sensory
metric (an internal representation of the geometry of
the external world) which could be the optic tectum
(see the scheme in Ref. 74), (2) a cerebellar motor
metric network that endows the executor mechanism
with a functional spacetime geometry (see the scheme
in Ref. 83), and (3) a sensorimotor transformation
that relates the two CNS geometries to one another;
e.g. by embedding one space, such as the sensory,
into another, such as the motor (see the scheme in
Ref. 76).
A tensorial interpretation of a particular sensori-
motor system yields a functional scheme (e.g. Fig. 1,
in Ref. 76). Such network schemes provide a math-
ematical interpretation of sensory processes, as yield-
ing covariant vectorial expressions, and motor pro-
cesses, as executing invariants with contravariant
components. Moreover, tensorial schemes can for-
cormally describe the nature of sensorimotor integration
as transforming a covariant vector, assigned to an
invariant expressed in the sensory frame, into a
contravariant expression in the motor frame. In
turn, the problem of co-ordination (the uniqueness of
a motor expression in an overcomplete executor-
mechanism) can be resolved as a covariant em-
bedding followed by a cerebellar motor transforma-
tion from covariant intention to contravariant
execution, even in the case when the covariant metric is
singular.\(^{74-76}\)
A tensorial interpretation of the cerebellum, which
is suggested to be the crucial final step of the sensori-
motor system, is shown in Fig. 2. The cerebellum is
featured as an "add-on" unit;\(^{75-77}\) the scheme in Fig.
2 demonstrates that a direct spinal cord pathway
could carry a motor intention-vector to represent the
motor output directly. Such approximative sensori-
motor transduction may have been an early evolu-
tionary "solution", where the directly obtainable but
incorrect motor vectors were used for motor execu-
tion. According to the "add-on" scheme, cerebell-
ectomy results in the direct execution of covariant
motor intention, through the down-going spinal
pathway shown schematically in Fig. 2. This feature
of the model corresponds to classical knowledge\(^{13}\)
that ablation of the cerebellum does not break the
sensorimotor transduction (and thus the cerebellum
cannot be considered as the organ implementing
this function); the ablation does result, however, in
a "dysmetric" motor activity decomposed both in
space and time.
The detailed operation of the essential cerebellar
circuit is described as follows. The covariant com-
ponents of the motor intention-vector, belonging to
a co-ordinate-system-invariant displacement (inv),
are shown in the upper left circle of Fig. 2. These
components are the orthogonal projections from the
invariant to each motor axis. Taking the motor
apparatus shown in Fig. 1 as a symbolic example,
these axes are 0, 25 and 37° with respect to each
other. Thus, a 5° physical invariant displacement,
The Function of the Essential Cerebellar Network

Covariant Intention to Contravariant Execution

Transformation

Fig. 2. Functional scheme of the essential cerebellar network: co-ordination by acting as a metric tensor, transforming covariant intention into contravariant execution. Sensorimotor transformation, by means of covariant-embedding procedure, yields independently and uniquely established projection-type intention components (even in case of overcompleteness) of an invariant (inv; upper left circle). This $i_k$ vector, if it directly descends (through the brain stem nuclei, bn) on the motor apparatus, would physically add to an incorrect execution (int; e.g. in acerebellar dysmetria). The role of the essential cerebellar network is to transform, by the “add-on” circuitry in the cerebellar cortex and nuclei, this intention to contravariant execution $e^e$ (exc; lower left circle). The metric-transformation is accomplished by the cerebellar corticonuclear network $g^{ik}$, by which the intention $i$, carried by mossy fibers (mf) to parallel fibers (pf) and to Purkinje cells (PC) is connected to cerebellar nuclear cells (cn). The inhibitory execution vector $e$, together with the mossy fiber collaterals to the nuclei, forms a corticofugal output $i-e$. This signal gives rise in the brain stem nuclei (bn) to the $e = i - (i-e)$ execution-vector output. Note that the olivary-climbing fiber system is not required for the essential cerebellar coordinative function.

with an arbitrary magnitude of 100, will yield a covariant vector $i_k = (100 94 85)^T$ where superscript $T$ denoted the transpose of the row-vector into column-vector. While these components do represent the displacement, their physical summation would yield a different displacement (int) with a different amplitude and direction (Fig. 2, upper left).

In contrast, the metric-type transformation through the cerebellar neuronal network yields contravariants that physically add to exactly yield the required invariant exc (Fig. 2, lower left inset). The three-segment limb with the $x$, $\beta$ and $\gamma$ system of co-ordinates therefore requires a contravariant metric-type transformer. This can be expressed numerically as $g^{ik}$, shown by a quantitative $3 \times 3$ matrix in Fig. 2. The system of connectivities, which implements such a matrix is the network between Purkinje and cerebellar nuclear cells (Fig. 2). This scheme conforms with the known cerebellar anatomy. Any $i_k$ covariant motor intention vector that enters the cerebellar cortex by mossy fiber activity will generate, through $g^{ik}$, the contravariant execution vector $e^e$ via cerebellar Purkinje cells (PC) in the cerebellar nuclei (cn). This vector $e^e = (80 25 -4)^T$ impinges on the cerebellar nuclear cells in an inhibitory form. Together with the excitatory mossy fiber collaterals into the nuclei, the cerebellar nuclei-cortical output will be $i_k - e^e = (20 69 89)^T$. In the brain stem nuclei (bn) this output will transform the intention vector into the required execution vector. This vector leaves the spinal relay nuclei as $e = i - (i-e) = (85 25 -4)^T$.

Note that the network essential to this covariant-contravariant transformation contains only mossy fibers, granule cells, Purkinje cells, cerebellar nuclear neurons and brain stem nuclear neurons. Strictly
speaking therefore, neither the climbing fiber system (and the inferior olive) nor the proprioceptive sensory mechanism is required for the explanation of the co-

ordinate function of the cerebellum (the covariant contravariant transformation) implemented by the “essential cerebellar network” (Fig. 2).76,77 As elaborated elsewhere,83 however, the climbing fiber system is essential in answering the question “how might such a network emerge?”

1.3 Problem: the genesis and modification of neuronal networks serving as covariant–contravariant metric-type transformers

The above exposition (point 1.2.3) of how the CNS could function by tensor-transformations assumes that the required matrices are in place, implemented by neuronal networks. A more profound question, however may be83 “how are neuronal networks organized such that they can embody and functionally support the necessary geometrical transformations?” While it is crucial to emphasize (as discussed in detail in Ref. 51) that the development of motor co-

ordination is not the function of the cerebellum (since the function is co-ordination), after identifying the function that cerebellar neuronal networks perform, answering the underlying developmental question may be very revealing from the point of view of emergence of neuronal networks in the CNS in general.

This major theoretical problem of network-organization can be readily illustrated by the cerebellum. Indeed the geometry of the motor apparatus is physically explicit and it is known that the matching functional geometry (implemented by the cerebellar neuronal network) develops from specific genetic and epigenetic arrangements which are expressed in the embriogenesis of the cerebellum.64,69,92 Once the basic co-ordination-function is performed by the emerged cerebellar networks, it is also known that a misalignment between the geometry of the execution system and a pre-existing functional motor geometry can result in a functional error. If this mismatch is minor, it may be absorbed by the overcompleteness of the functional transformation.83 For misalignments of the geometries that follow a certain trend (e.g. when motor co-ordination has to keep up with the increase in body size during growth) the cerebellar system must respond with a degree of adaptability, just as in every subsystem of the CNS.51 It is also known that major discrepancies may trigger realign-

ment of the internal and external geometries—an important function that allows the organism to re-

sume an optimal interaction with the surrounding world (cf. the reversal of the vestibulo-ocular reflex by prisms and the ensuing compensation).23 As sug-

gested in the three-step scheme of sensorimotor trans-

formation,76,83 the neuronal networks implementing a sensory metric tensor and a motor metric are the means of incorporation of such functional geometries. Therefore the adaptability of the whole sensorimotor operation raises the question how a limited degree of adaptibility of each of these metrics may contribute to a maintained match of the external physical and internal functional geometries. As for the cerebellum, the assumed position-dependent motor metric-function implies, for example, that the neuronal network must undergo constant phasic updating by the climbing fiber system in order to perform the required non-Euclidean (non-constant) metric function.

While the question of emergence of neuronal net-

works, acting as metric tensors, can be conveniently approached at the level of motor co-ordination, this question is more profound than a limited study of sensorimotor operations implies. Indeed one of the fundamental challenges in neuroscience is that of providing a formal account of the ability of geomet-

eties, intrinsic and extrinsic to CNS, to organize one another so that a set of optimally interactive geometries can evolve.

2. METAORGANIZATION OF CNS GEOMETRIES

2.1. The principle of metaorganization

In search of the principles by which neuronal networks could be organized, one needs to rely on the power of both the mathematical formalism and of the biological insights derived from direct experimental acquaintance with the problem.

2.1.1. Optimal mathematical characteristics of the procedure of establishing metric networks: iterative algorithm for eigenvector-expansion. A general prin-

ciple in all biological systems appears to be that their structure is parallelly organized and distributed and their function develops by iterative procedural means. An example is the acquisition of cerebellar temporal lookahead by the implementation of a type of Taylor series expansion.80 Thus, when addressing the questions of the distributed implementations of neuronal metrics, one may consider the manner in which matrices, acting as metric tensors, can be established by iterative reverberative procedures in a form of matrix-expansion.

Another lead is provided by the fact that the primary entities in CNS function are the covariant sensory and contravariant motor expressions, and not the metrics that may or may not connect them. The co- and contravariant inter-relations evoked by external physical reality can manifest themselves in case of a lack of an explicit realization or even in case of a total absence of an ordinary metric: both sensory and motor processes are possible without an inter-

mediate co-ordinated transfer. Any metric expression is therefore secondary, as it derives from a process by which given co- or contravariants may be converted into one another.

As for a matrix acting as a metric, such matrix is symmetrical—nd determined by those co- and contra-

variant vectors which constitute eigenvectors. The
metric is indeed fully characterized by these special input–output vectors that are identical in their normalized form.91

The above two mathematical considerations provided the impetus for pointing to the steady-state covariant–contravariant reverberation as the key for a formal geometrical characterization of the function and emergence of neuronal networks. An eventual identity of covariant sensory information and contravariant motor output (where the input can directly determine the output in the form of an eigenvector without the necessity of an interconnected metric transformation) provides the basis of metaorganization.

2.1.2. Optimal characteristics for the biological procedure for establishing metric networks: tensorial interpretation of tremor. The formative aspects of oscillatory behavior are most conspicuous during embryogenesis.69 It is known, for example, that embryos evolve through characteristic tremor and oscillatory twitching.68 Reverberative resonance is therefore a dominant characteristic which may reveal fundamental properties of the movement effector and the functional properties of related neuronal networks. Oscillations have also been analysed in detail by numerous workers both from biological and mathematical points of view (see Refs 4, 5, 12, 22, 24, 28, 60, 68 and 100).

The most significant feature of tremor may be expressed formally by assuming that the proprioception system provides a covariant measure of the contravariant motor execution. Indeed it has been suggested7 that musculoskeletal systems endowed with proprioceptive feedback, where the frame of reference both for motor execution and sensory reception is common, can base their function on the following fact. The contravariant motor action and the covariant proprioception, belonging to the same physical invariant, represent together the covariant metric tensor. That is, for any given contravariant execution vector e = (1 0 0)T, where superscript T denotes the transpose of the row-vector into column-vector. These covariant signals are identical, i.e. when they both constitute an eigenvector of the system.

The above two mathematical considerations provided the impetus for pointing to the steady-state covariant–contravariant reverberation as the key for a formal geometrical characterization of the function and emergence of neuronal networks. An eventual identity of covariant sensory information and contravariant motor output (where the input can directly determine the output in the form of an eigenvector without the necessity of an interconnected metric transformation) provides the basis of metaorganization.

2.1.2. Optimal characteristics for the biological procedure for establishing metric networks: tensorial interpretation of tremor. The formative aspects of oscillatory behavior are most conspicuous during embryogenesis.69 It is known, for example, that embryos evolve through characteristic tremor and oscillatory twitching.68 Reverberative resonance is therefore a dominant characteristic which may reveal fundamental properties of the movement effector and the functional properties of related neuronal networks. Oscillations have also been analysed in detail by numerous workers both from biological and mathematical points of view (see Refs 4, 5, 12, 22, 24, 28, 60, 68 and 100).

The most significant feature of tremor may be expressed formally by assuming that the proprioception system provides a covariant measure of the contravariant motor execution. Indeed it has been suggested7 that musculoskeletal systems endowed with proprioceptive feedback, where the frame of reference both for motor execution and sensory reception is common, can base their function on the following fact. The contravariant motor action and the covariant proprioception, belonging to the same physical invariant, represent together the covariant metric tensor. That is, for any given contravariant execution vector e = (1 0 0)T, where superscript T denotes the transpose of the row-vector into column-vector. These covariant signals are identical, i.e. when they both constitute an eigenvector of the system.

The above considerations lead to the metaorganization principle, summarized in the following.

1. The eigenvectors of the covariant metric of a motor system can be established by reverberations, resulting from the return of covariant proprioception to motor effectors as if they were contravariants. (2) This oscillation will reach a steady-state of covariant–contravariant eigenvectors. (3) These eigenvectors and eigenvalues (or the generalized inverses of the latter) can be used to generate either a duplicate or a complementer of the covariant metric tensor. (4) The resulting metric-type networks (e.g. that of the cerebellum) can be used as an internal function representation of an external geometry (e.g. used for motor co-ordination).

It is important to emphasize that co-ordination established by metaorganization may not be necessary for some stereotyped movements (especially those in lower vertebrates). Operations such as basic locomotory or grooming actions do not represent the class of co-ordinated goal-oriented movements that are the focus of this study. Rather they seem to be "preprogrammed" activities based on fixed-pattern generators at spinal cord level, for which a higher cortical involvement of the intention, or a cerebellar co-ordination of the execution, is not necessary.

2.1.3. Reverberation of proprioceptive covariants as contrariants, in order to establish the eigenvectors implicit in the physical geometry of the motor apparatus. The interpretation (in section 1.2) of the proprioception signals and motor signals as covariants versus contrariants provides a concrete example of the physical implementation of the metaorganization of CNS. The proposed procedure will be shown to yield a co-ordinated control of an overcomplete muscular system. The implementation is based on the reverberation of the covariant proprioception-afferent as if it were a contravariant-executor motoneuron effferent. This proprioception-execution reverberation will set up an oscillation of the motor apparatus that reaches a steady-state when the input and output signals are identical, i.e. when they both constitute an eigenvector of the system.

The procedure shown in Fig. 3, utilizes the motor system of Fig. 1 and the cerebellar neuronal network in Fig. 2 with an arbitrary vector descending the motor executor system. It is emphasized that this initial vector could be arbitrary and may arise from internal "noise" of the circuits. Let the motoneurons innervating the pectoralis muscle produce a burst of spikes of unitary strength which results in a movement of the hand along the z direction. This arbitrary motor signal can be described as a contravariant vector e = (1 0 0)T, where superscript T denotes the transpose of the row-vector into column-vector. According to section 1.2.2, the peripheral sensors, which measure the tensions in the tendons, are capable of yielding the covariant sensory components of the generated invariant. These components arise geometrically by establishing the orthogonal projection components of the unitary motor vector e (along direction z, to the β and γ axes). These covariant components can be calculated from the contravariant vector by multiplying it with the covariant metric tensor of the motor system, which is simply the table of cosines among the axes.77 The local system of coordinates for the movements of the index finger is shown in Fig. 1, by axes at 185, 160 and 148° angles, in the two-dimensional physical plane, where the angles are measured from the customary right-horizontal 0° direction. Thus the covariant metric of
the physical motor apparatus is:
\[ \mathbf{g}_{\text{out}} = \cos(\phi_{\text{out}}) \]
\[ = \begin{bmatrix}
\cos(185-150) \cos(185-160) \cos(185-148) \\
\cos(160-150) \cos(160-160) \cos(160-148) \\
\cos(148-150) \cos(148-160) \cos(148-148)
\end{bmatrix}
= \begin{bmatrix}
1.000 & 0.906 & 0.799 \\
0.906 & 1.000 & 0.978 \\
0.799 & 0.978 & 1.000
\end{bmatrix} \] (3)

Given that such a covariant metric of the motor execution is represented by a matrix where the components are the cosines of the angles among the axes, two properties of the matrix are given: (1) the matrix is symmetrical, since \( \phi_{\text{out}} = \phi_{\text{in}} \) and (2) its components are real values since \( \cos(\phi) \) is a real number for any angle. Symmetrical real-valued matrices constitute a special subclass of Hermitian matrices\(^8\) and thus are characterized by having a set of orthogonal eigenvectors with eigenvalues that are real numbers.\(^11\) Thus an actual physical procedure to generate such metric matrices is possible; these are implemented by reverberation, as shown in the remaining part of this section.

The numerical example, shown in Fig. 3, starts with the initial arbitrary contravariant motor execution vector \( \mathbf{\epsilon} = (1 \ 0 \ 0)^T \). The corresponding first covariant expression, detected by proprioception, will therefore be \( \mathbf{p} = (1.000 \ 0.960 \ 0.799)^T \). One can establish these values either geometrically, by taking the orthogonal projection to the other axes of the non-zero component of \( \mathbf{\epsilon} \), or by computing its components through the covariant metric according to \( \mathbf{p} = \mathbf{g} \times \mathbf{\epsilon} \). It is assumed that \( \mathbf{p} \) is normalized before it is reverberated as if it were contravariant (normalized vectors are denoted by barred symbols). This vector will descend on the motor system as \( \mathbf{\epsilon}' = (0.638 \ 0.578 \ 0.509)^T \). This contravariant output will then produce an arm position that will be measured covariantly in the second reverberation by the proprioceptors, which detect tension in the tendons (\( \mathbf{g} \)). Application of the covariant metric reveals that \( \mathbf{p}' = (1.568 \ 1.654 \ 1.584)^T \). If this proprioceptive vector is reverberated for the second time in normalized form as \( \mathbf{p}'' \), it will be executed as \( \mathbf{\epsilon}'' = (0.565 \ 0.596 \ 0.571)^T \). This vector will then be covariantly measured as \( \mathbf{p}'' = (1.561 \ 1.666 \ 1.605)^T \) and reverberated in the third cycle in a normalized form as \( \mathbf{\epsilon}''' = (0.559, 0.597, 0.575)^T \). As verified through the application of this vector to the covariant metric, the third sensory proprioception will yield \( \mathbf{p}''' = (1.560 \ 1.666 \ 1.606)^T \). Repeating the cycle, the fourth contravariant return will be \( \mathbf{\epsilon}'''' = (0.559 \ 0.597 \ 0.575)^T \). Note that the above reverberation stabilizes with a vector that is identical in its proprioceptive and execution forms. The reverberated signals are identical after the third cycle: \( \mathbf{\epsilon}'''' = \mathbf{p}'''' = \mathbf{p} = (0.559 \ 0.597 \ 0.575)^T \).

The example of reverberation shown in Fig. 3 demonstrates that after a rapid convergence the oscillation of the system reaches a steady-state of the eigenvectors. Without such normalization, during the reverberation each component of the \( n \)th covariant sensory vector would be \( \lambda = 2.791 \) times greater than the corresponding \( n \)th contravariant motor vector component:
\[ \mathbf{p} = \lambda \times \mathbf{\epsilon} = 2.791 \times \mathbf{\epsilon} = 2.791 \times (0.559 \ 0.597 \ 0.575)^T \] (4)

The factor \( \lambda \) is known as the eigenvalue.\(^8\) It can be measured by the same operation as the normalization since \( \lambda \) is the necessary degree of change of the magnitude (normalization) of the vector before its reverberation.

Such a covariant-contravariant pair, given above in normalized form as \( \mathbf{\epsilon} = \mathbf{p} = \mathbf{E}_1 = (0.559 \ 0.597 \ 0.575)^T \), where the covariant and contravariant forms differ by only a constant coefficient for each component, is called an eigenvector of the covariant metric in the given frame of reference and the \( \lambda_1 = 2.791 \) constant is the first eigenvalue belonging to the first eigenvector \( \mathbf{E}_1 \).

The above iterative mathematical method of finding an eigenvalue and the belonging eigenvector is widely used in computer science in the case of large symmetrical real-value matrices as in Hermitian matrices (cf. 7.27 in Ref. 11). Although the eigenvector-decomposition of matrices was not applied to the metric tensor and the co- and contravariant tensorial aspects had not been recognized, the neuro-biological significance of such decomposition has been greatly exploited (see, for example, Ref. 5).

The utilization of the eigenvectors found by reverberation for the genesis of a metric-type network is illustrated in Fig. 4. The most crucial step of the metaorganization-process is the prevailing of the found eigenvector to a cortico-nuclear array of neurons, both directly (e.g. via climbing fiber collaterals to the cerebellar nuclei) and indirectly (e.g. via climbing fibers to the Purkinje cells which in turn project to the nuclei). Such convergence of the same (climbing fiber) vector may imprint an array of neurons by the dyadic (outer) product of the vector with itself. Such a dyad \( \mathbf{D}_1 = \mathbf{E}_1 \times < \mathbf{E}_1 \) (symbol \( > \) denotes the outer product of vectors) can be seen both in the connectivity diagram and also numerically (Fig. 4). The dyadic product of an eigenvector with itself will be called an “eigendyad”. As shown, \( \mathbf{D}_1 \) will serve as the first approximation of \( \mathbf{g}^\bullet \), denoted by \( \mathbf{g}^* \).

Once the first eigenvector and corresponding eigenvalue is established, the remaining eigenvectors of the system can be found by reverberating a vector whose direction is orthogonal to that of the previously found eigenvectors. Reverberation can therefore proceed by filtering out from \( \mathbf{p} \) before every reverberation, the already found eigenvector-component \( \mathbf{p}_r = (\mathbf{E}_r > < \mathbf{E}_r) \times \mathbf{p} \), (see Fig. 4 in Ref. 11):
\[ \mathbf{p} - \mathbf{p}_r = \mathbf{p} - \sum_{n} (\mathbf{E}_m > < \mathbf{E}_m) \times \mathbf{p}. \] (5)
The Function of the Proprioception-Motor Execution Spinal Circuit

Establish Eigenvectors via Subcortical Reverberation

Subcortical reverberatory circuit

Cortical circuits

Abbreviation

Fig. 3. Oscillatory reverberation of motor execution-proprioception, establishing the eigenvectors of the motor apparatus. (A) Subcortical reverberatory circuits. Ascending spinocerebellar pathways carry covariant proprioception (p) of any execution (e), via mossy fiber (mf) collaterals into the cerebellar nuclei (cn). The reverberation-loop closes on the brain stem nuclei (bn), with descending motoneuron pathways, carrying execution components that will be physically assembled as contravariants (c). The covariant metric $g_{nk}$, inherent in the physical geometry of the motor apparatus, will provide for any contravariant executor vector its proprioceptive covariant counterpart. (B) A quantitative example for the stabilization of the execution (e)-proprioception (p) reverberation in the eigenvector ($E_1$). Barred vectorial symbols denote normalized vectors. Starting with an arbitrary $e = (1 \ 0 \ 0)$ execution, already after the second reverberation the execution and proprioception vectors are identical, meaning that an eigenvector is established.

Without this filtering the reverberation would again converge to the same eigenvector; while the filtering forces the reverberation to be confined to the direction orthogonal to the previously found eigenvectors. The above iterative computation technique of the eigenvectors and eigenvalues is possible since in real-valued symmetrical matrices the eigenvectors are mutually orthogonal and the eigenvalues are real (7.27 in Ref. 11 and theorem 4-4 in Ref. 87).

Similarly it can be calculated that the above mathematical but physically implementable method of filtered reverberations leads to the second normalized eigenvector and the corresponding eigenvalue:

$$E_2 = (-0.783 \ 0.153 \ 0.603)'; \ \lambda_2 = 0.209.$$  

The end of the reverberative iterative search, indicating that all the eigenvectors and eigenvalues have been found, can be determined as follows. In a physical process it can be monitored when the reverberating vector, filtered for all previously found eigenvectors, becomes zero. At that point no more eigenvectors can be found and thus the search stops. In a mathematical process, when the covariant metric can be made explicitly available by numerical calculation, the end-point of the reverberative search can also be determined by comparing the sum of found eigenvalues to the trace (tr) of the matrix of the covariant metric, which is defined as the sum of the diagonal elements. Since

$$\text{tr}(g_{nk}) = \text{tr}(g^{ek}) = \sum_n \lambda_n,$$

in the above case

$$\text{tr}(g_{nk}) = \text{tr}(g^{ek}) = 1 + 1 + 1 = 2.791 + 0.209.$$  

In our example this means that only two non-zero eigenvalues can be found and therefore the search can
end. The existence of only two non-zero eigenvectors in a three-matrix reflects the fact that the three-axis frame of reference is overcomplete compared to the two-dimensional space. Since the eigenvectors of a symmetrical real-valued matrix are mutually orthogonal, when the motor apparatus shown in Figs 1–3 is confined into a two-dimensional plane, only a second orthogonal vector can be found in addition to the first direction determined by the eigenvector.

2.1.4. Spectral representation of the covariant metric tensor and its proper inverse (or Moore–Penrose generalized inverse) as expressed by their eigendyads; the outer products of eigenvectors weighed by the corresponding eigenvalues. The covariant metric tensor was established in (3) by calculation as a matrix composed of the cosines among co-ordinate axes. In the physical motor mechanism this metric is only implicitly available in the sense that for every particular contravariant motor execution vector the physical effector system provides its covariantly measured proprioceptive vectorial counterpart. However, with the use of the eigenvectors and eigenvalues found by the above physical oscillation the metric tensor can be made explicit either in its co- or contravariant form. Accordingly a neuronal network can be constructed that implements the matrix which establishes functional geometries, e.g. the transformation from covariant motor intention into contravariant motor execution which was proposed as the basis for motor co-ordination.11

The method of constructing the metric is based on the spectral representation of the covariant metric6 (cf. p. 132, theorem 7.3 in Ref. 11, or theorem 8.8 in Ref. 91):

$$\mathbf{g}_{ab} = \sum_m \lambda_m \mathbf{E}_m \times \mathbf{E}_m$$

where \(\mathbf{E}_m\) is the \(m\)th normalized eigenvector and \(\mathbf{E}_m \times \mathbf{E}_m\) is the outer (dyadic) matrix product of the \(m\)th eigenvector.

The Function of the Olivo-Cerebellar Climbing Fiber System

Metaorganization (Genesis & Modification) of Functional Geometry

Fig. 4. Genesis of the cerebellar corticonuclear metric via metaorganization, by means of imprinting the dyads of eigenvectors found by reverberation. The reverberatory scheme shown in Fig. 3 is supplemented by the olivary system (IO), which compares the ascending proprioception (p) and descending execution-vector (e). (1) Having detected an identity, an eigenvector (E) is found, that is implemented in the olive. (2) The eigenvector is utilized to generate the corticonuclear network, by being transmitted, via climbing fiber vector (c = E) to both the Purkinje cells and the cerebellar nuclear cells. (a) The dyad of the eigenvector (eigendyad) \(\mathbf{D}_1\) is shown numerically, yielding the first approximation of the metric \(\mathbf{g}_{nk}\); in effect determining the principal axes of the cerebellar tensor ellipsoid (cf. Fig. SC). (b) The filtering of the reverberation. After having found the first eigenvector, the already established \(\mathbf{D}_1\) dyad serves as a filter that removes the \(\mathbf{p}_F = (\mathbf{E}_1 \times \mathbf{E}_1)\mathbf{p}\), forcing the reverberation-vector \(\mathbf{p} - \mathbf{p}_F\) to be orthogonal to the already established eigenvector.
In the given example the above formula yields the spectral representation of $g_{\kappa\ell}$ as follows:

$$g_{\kappa\ell} = 2.791 \begin{bmatrix} 0.559 & 0.559 & 0.597 & 0.575 \\ 0.575 & -0.783 & -0.783 & 0.153 \\ 0.513 & 0.334 & 0.321 & -0.783 \\ 0.321 & 0.344 & 0.331 & 0.153 \end{bmatrix} + 0.209 \begin{bmatrix} 0.613 & -0.120 & -0.472 \\ 0.120 & 0.023 & 0.092 \\ -0.472 & 0.092 & 0.364 \\ 0.334 & 0.356 & 0.343 \end{bmatrix}$$

(9)

$$= 2.791 \begin{bmatrix} 0.559 & 0.559 & 0.597 & 0.575 \\ 0.312 & 0.334 & 0.321 & 0.312 \\ 0.344 & 0.344 & 0.331 & 0.344 \end{bmatrix} + 0.209 \begin{bmatrix} 0.999 & 0.907 & 0.800 \\ 0.907 & 0.999 & 0.976 \\ 0.800 & 0.976 & 1.000 \end{bmatrix}$$

(10)

The resulting covariant metric is, with practical precision, identical to the one computed directly from the cosines. The computation was actually performed for 7 decimal digits; however, as shown here, results come within ±0.002 precision even if the calculation is rounded to the biologically relevant 3 digits.

The two most important aspects of metaorganization are that the above iterative procedure which yields the spectral representation of the covariant metric (a) is established by a physically executable oscillation which is set up simply by a recurrent reverberation and (b) will yield not just the covariant metric tensor itself, but also either its proper inverse (if it exists) or its Moore-Penrose generalized inverse (in case of overcompleteness). The former applies if the space is complete (e.g. it is Riemannian) and thus the inverse of the covariant metric tensor exists. The latter applies if the covariant metric is singular and thus the space is non-Riemannian. The unified expression of the proper, or generalized metric, is:

$$(g_{\kappa\ell})^+ = \sum_m \lambda_m^+ \times (E_m > < E_m)$$

(11)

where $\lambda_m^+$ is the generalized inverse of the $m$th eigenvalue (3.6.2. in Ref. 1).

$$\lambda_m^+ = \begin{cases} 1/\lambda_m & \text{if } \lambda_m \neq 0 \\ 0 & \text{if } \lambda_m = 0. \end{cases}$$

(12)

For further details of the mathematics of generalized inverses and the Moore-Penrose pseudoinverse see Refs 1 and 9. For its introduction into tensor network theory of the CNS see Refs 74–78. For non-tensorial neurobiological applications of the generalized inverse see Ref. 41 and for robotics see Ref. 40. It is emphasized that in the metaorganization algorithm (a) the spectral decomposition is applied not to any matrix, but specifically to the covariant metric tensor, and (b) the eigenvector, established via an oscillation, is used to generate a generalized inverse of the covariant metric, expressed not in Cartesian but in non-orthogonal co-ordinates.

In the given numerical example, the above formula of Moore-Penrose-generalized inverse of the covariant metric yields:

$$g_{\kappa\ell}^+ = 1/2.791 \begin{bmatrix} 0.559 & 0.559 & 0.597 & 0.575 \\ 0.575 & -0.783 & -0.783 & 0.153 \\ 0.312 & 0.334 & 0.321 & -0.783 \\ 0.321 & 0.344 & 0.331 & 0.153 \end{bmatrix} + 1/0.209 \begin{bmatrix} 0.613 & -0.120 & -0.472 \\ 0.120 & 0.023 & 0.092 \\ -0.472 & 0.092 & 0.364 \\ 0.334 & 0.356 & 0.343 \end{bmatrix}$$

(13)

and, if the dyads of eigenvectors ("eigendyads"), weighted by the eigenvalue are explicitly calculated, is equal to

$$g_{\kappa\ell}^+ = \begin{bmatrix} 0.112 & 0.120 & 0.115 & 0.120 & 0.127 & 0.123 & 0.115 & 0.123 & 0.118 \\ 0.293 & -0.574 & -2.259 & -0.574 & 0.110 & 0.440 & -2.259 & 0.440 & 1.742 \\ 3.045 & -0.454 & -2.114 & -0.454 & 0.237 & 0.563 & -2.144 & 0.563 & 1.860 \end{bmatrix}$$

(14)

The verbal expression of the above is the following. The Moore–Penrose-generalized inverse of a matrix, that conserves the eigenvectors of the original matrix, is constructed as the sum of dyadic outer product of each eigenvector with itself, i.e. of the dyads weighted by the (generalized) inverse of the corresponding eigenvalue.

Note that the Moore–Penrose-generalized inverse of the covariant metric tensor has already been numerically calculated by applying the metaorganization principle and algorithm as proposed earlier. The numerical example of Fig. 2, using the metric type neuronal network, has also been shown with the Moore–Penrose-generalized inverse components of the contravariant metric. Nevertheless, an exposition of the reverberative procedure has not hitherto been offered. The metaorganization principle is elaborated here as an oscillatory procedure in accordance with the classic notions on recurrent reverberating circuits; the theory of closed “self re-exciting” chains of neurons, the control-theoretical emphasis on “feedback and oscillation” in Chap IV of Ref. 100, the emphasis on reverberation in Ref. 12 and the recent analysis of the central role of motor oscillations at the neuronal level in motor development.

2.2 Conceptual interpretation of the principle of metaorganization of neuronal networks

A basic interpretation of the principle of metaorganization relies on the fact that the orthogonal
spectral decomposition of the matrix of the covariant metric is conceptually equivalent to viewing the motor-transformation through the metric not as a wholly integrated operation, but as composed of transformations through separable eigendyads. As shown each eigendyad is the outer product of a normalized eigenvector with itself, the eigenvalue serving as a coefficient. It can easily be verified that an eigendyad transforms an eigenvector into itself, which will only be stretched or shortened by the eigenvalue coefficient. Since the eigenvectors are mutually orthogonal, each eigendyad operates only on that vector component which lies in its own direction; it is "intransparent" (producing zero output) to components that are diagonal to it. In the metric transformation of contravariants into covariants the magnification coefficient is the eigenvalue of the covariant metric. This explanation that the reverse contravariant metric-type transformation (via the generalized inverse of the covariant metric) must be performed through the same set of eigendyads (serving as the eigenvector-transformers), while the coefficient of each dyad must be the generalized inverse of the eigenvalue.

It must be emphasized, however, that this decomposition into independent "channels" of amplification (for biological correlates of such channels, see Ref. 47) is only possible along the mutually orthogonal eigenvectors of the system. Therefore, a customary interpretation of a horizontal eye movement, for example, as a separable direction from vertical and torsional eye movements may be improper. The eigenvectors of the oculomotor-covariant metric (a) have not even been established at the time of such customary interpretation and (b) when they have recently been calculated, they turned out to be greatly different from the horizontal direction (by about 45°, cf. Ref. 77). Thus some of the most immediate experimental paradigms, derived from the proposed metaorganization principle are (a) to establish experimentally the eigenvectors in biological systems and (b) to determine if the amplification of co- to contravariant vector-components can be independently altered along the mutually orthogonal eigenvectors and interdependently along all other directions, as suggested here by this theory.

A more abstract conceptual interpretation of the metaorganization principle is possible by a graphic depiction of the function of the cerebellum, as a geometrical distortion which is implemented by the covariant metric and its generalized inverse. Such is possible in the form of a tensor-ellipsoid (see Fig. 5C, after Ref. 78). The covariant-contravariant transformation (and vice versa) is visualized in Fig. 5(C) as a geometrical distortion of an ellipsoid of the cerebellar input intention vectors $\mathbf{p}_i$ into a circle of execution vectors $e^i$. Such transformation is determined by the principal direction-axes of the ellipsoid (given by the eigenvectors) and by the magnitude-distortion (where the lengths of the principal axes along each eigenvector correspond to the eigenvalue).

This geometrical definition of the primary cerebellar function leads one to the secondary question of the development of the function through the emergence of the neuronal network that implements this transformation. While the geometrical symbolism in Fig. 5(C) provides a concise interpretation of the function itself in the first place, secondarily it also suggests that its development may be determined by the double procedure of (1) establishing and storing of the eigenvectors of the tensor-ellipsoid and (2) trimming the eigenvalues (i.e. adjusting the principal axes to their proper lengths). The steps by which these tasks are accomplished are reviewed next.

2.3 Elaboration of the metaorganization principle, explaining the genesis and modification of cerebellar metric-type neuronal networks

In section 1.2.3, it was summarized and quantitatively demonstrated how a covariant-contravariant transformer-matrix can serve as a cerebellar motor coordinator. In section 2.3.1 below, a concrete numerical example is given for how the metaorganization principle can be implemented by a process in the CNS to generate such cerebellar networks.

2.3.1. The genesis of functional geometries as implemented by neuronal networks. The summary diagram in Fig. 5 (after Refs 75-78) illustrates a general scheme of the function, genesis and modification of the cerebellar networks. By including Fig. 2, it shows that the function of the essential cerebellar network can be interpreted as performing a covariant-contravariant transformation. However, Fig. 5 also indicates that this network converts the motor intention-signals into motor signals, taking in account not only space co-ordinates as shown in Fig. 2, but also spacetime co-ordinates as shown in Ref. 83. The "stacks" of Purkinje cells, which serve as "temporal lookahead-modules",80 each model requiring about two hundred cells, are illustrated by a schematic triad of Purkinje neurons (Fig. 5D).

The illustration shown in Fig. 5, also encompasses the circuitry necessary for the establishment of the eigenvectors by reverberation and the network-approximation of the metric by its eigendyads (see Figs 3 and 4). The covariant proprioception vectors, $p$, enter to the cerebellum via mossy fibers that give collaterals to the cerebellar nuclei. Because the Purkinje cell-cerebellar nuclear cell synaptic connectivity is established late in embryogenesis, we propose that reverberation specifies this connectivity in an epi-genetic manner. Thus, initially for any proprioceptive input, the mossy fiber input to the cerebellar cortex will yield a zero vector through the Purkinje cells at the cerebellar nuclei before the metaorganization process is implemented. As a result the nucleofugal output will carry the same information as the mossy fiber input itself. This output is then introduced in the
Fig. 5. Co-ordination by the cerebellar networks, and their genesis and modification by metaorganization. (A) Metaorganization-algorithm for the genesis of network-matrix that approximates the metric $g^a_k$ by the dyads of its eigenvectors; $D_n$, via climbing fiber vector, carrying the eigenvector; $c = E_a$. Such algorithm sets up a tensor-ellipsoid with principal axes of the eigenvectors (cf. Fig. 5C). (B) Metaorganization-algorithm for the modification of the network matrix, in order to correct the eigenvalues of the tensor-ellipsoid. Geometrical inset diagram illustrates, by means of a simplified two-dimensional frame, that a goal (G), given by intention-components ($i_{int}$) is improperly executed if the existing eigenvalue is incorrect. Thus, the execution-components ($e_{exe}$) add to an erroneous performance-point (P), that is covariantly relayed back by perception ($p_{perc}$). Graph demonstrates, that by the projections of points G and P to the established eigenvectors ($E_k$) the difference of the existing and the desirable eigenvalues can be measured. Thus, a correction-vector (c) is established by the olive, so that climbing fibers imprint a dyad of correction ($\Delta g_{ak}$). (C) Geometrical representation of the function of the essential cerebellar network as a covariant intention to contravariant execution transformer. A circle of execution-vectors (exc), when expressed in the form of intentions (int), would be distorted into an ellipse. This tensor ellipsoid is determined by its eigenvectors ($E_{1,2}$) and eigenvalues ($\lambda$). Thus the function of the cerebellum is symbolized as a geometrical "mirror-like" transformation of distortion-prone intentions into proper execution. (D) Composite diagram of circuits necessary for the essential function, plus its genesis and modification. Proprioceptive signals are shown in green, intention in blue, execution in red and correction in yellow. The essential function is implemented by the blue-to-red spacetime metric circuit ($g_{ak}$), the genesis is implemented by the green-to-red reverberation and imprinting (via the yellow circuit) of the corticonuclear network. The olivary system (IO; yellow) subserves modification by relying on the green proprioceptive, blue intention and red execution signals. Temporal "lookahead-module" of stacks of Purkinje cells symbolize that the metric transformation is not restricted to the space domain, but applies to a unified spacetime manifold.
A **Genesis**

Principal Axes of Tensor Ellipsoid Established by Generating Corticonuclear Dyads of Eigenvectors, Found by Proprioception Execution Reverberation

\[ c = E_j \]

\[ \tilde{g}^{nk} = \sum D_j = \sum E_j \rangle \langle E_j \]

B **Modification**

Curvature of Tensor Ellipsoid Modified, Based on Goal-Performance Error, via Eigenvalue-Correction by CF Dyad

\[ c = \sum (\Delta \lambda_j)^{1/2} \cdot E_j = \sum \left( E_j \cdot \frac{e_j}{E_j \cdot p} - E_j \cdot \frac{e_j}{E_j \cdot i} \right)^{1/2} \cdot E_j \]

\[ \Delta g^{nk} = \sum \Delta \lambda_j \cdot (E_j \rangle \langle E_j) \]

C **Function**

Covariant Intention - Contravariant Execution via Essential Network

\[ \lambda_1 \cdot 2.7 \]

\[ \lambda_2 \cdot 0.2 \]

D **Network**

Spacetime Metric

Intention

Proprioception

Lookahead Module

Correction

Execution

Proprioception
motor system as if it were a contravariant effector vector, with only a signal-reversal at the brain stem nuclei, thus leading to stabilizing oscillations.

The imprinting of the eigendyads into the cerebellar corticonuclear circuitry is illustrated in Figs 4 and 5(A). In the scheme shown in Fig. 5 small additional circuits are necessary for normalizing the amplitude of the reverberation and for identity detection to monitor the eigenvalue stabilization when \( \mathbf{\tilde{E}} = \mathbf{p} \). Both operations can easily be accomplished by taking their inner product. This can be accomplished by introducing an interneuron (see below). Indeed, in order to normalize the amplitude, the inner product is initially obtained by multiplying \( \mathbf{p} \) with itself. With this factor the local inhibitory interneuron can reduce the magnitude of the vector. This operation corresponds to the amplitude-stabilization by Golgi cells proposed earlier. Here the effect of the Golgi inhibition is normalization, with the firing rate of the Golgi cell being the measure of the eigenvalue. In order to monitor the degree of eigenvector-stabilization, for the inner product of \( \mathbf{\tilde{p}} \) and \( \mathbf{\tilde{E}} \) the interneuron is organized such that it will only reach a unitary firing rate when \( \mathbf{p} \) and \( \mathbf{E} \) are eigenvectors.

Such a simple interneuron circuit can determine whether the convergence of the inner product of the two normalized vectors is close enough to (1.00 in absolute value) to indicate that an eigenvector has been found. At that point both the eigenvalue, automatically provided by the normalizer, and the corresponding eigenvector (taken either from the ascending or descending pathway) are available for constructing the eigendyads as shown in section 2.1.4. Such imprinting requires (a) a convergence of the identical eigenvector on both the row and column elements of a matrix in order to establish their product and (b) that ionic mechanisms are capable of triggering chemical changes at the postsynaptic element which may modify intrinsic electroresponsiveness in a manner proportional to this product. The mossy fiber–parallel fiber–Purkinje cell system and the climbing fiber–Purkinje cell system could in principle be capable of generating such modification, since both these pathways carry the eigenvectors at such steady-state of the oscillation. Still, several considerations support the argument that such an adaptive modification may occur through the corticonuclear synaptic network. First the inferior olive signal is received directly from olivocerebellar collaterals arriving at the nuclei and indirectly via the climbing fiber activation of Purkinje cells, so that a convergence capable of evoking the required integration can occur. Second, modification of the cerebellar nuclei is consistent with the finding that vestibular adaptation is retained after ablation of most of the cerebellar cortex. Third, as will be seen in section 2.3.2, in pathological conditions where a modification of the corticonuclear circuitry after its initial genesis may be required, the direct and indirect climbing fiber vectors facilitate an adaptive change at the site of their convergence, presumably in the cerebellar nuclei.

During the process of imprinting the eigendyads into the corticonuclear cytoarchitecture, as proposed recently, the inferior olive would store the eigenvectors and eigenvalues found by reverberation. Such storage will be required for the conformation of the genesis; the ongoing modification of the network. The suggestion that the olive has storage properties is particularly apt in view of the intrinsic capacity of olivary and related neurons for rebound oscillation. Long-term ionic conductance-change mechanisms, consistent with the proposed storage, have been demonstrated experimentally. According to this view, following the storage of the \( m \)th eigenvector \( \mathbf{g} \) the olive would signal a climbing fiber vectorial correction \( \mathbf{c} = \mathbf{E}_m \), which would modify the corticonuclear integrative properties by an additive

\[
\mathbf{g}_m = \mathbf{g}_{\text{old}} + \mathbf{g} = \mathbf{g}_{\text{old}} + \left( \mathbf{c} > < \mathbf{c} \right) = \mathbf{g}_{\text{old}} + \left( \mathbf{E}_m > < \mathbf{E}_m \right) \quad (15)
\]

thereby imprinting the actual eigendyad. Using the first eigenvector–eigenvalue, it can be easily verified that such corticonuclear convergence of climbing fiber vectors will yield the first eigendyad shown in 2.1.4. The result of this procedure will be an approximation of the metric by its eigendyads:

\[
\mathbf{g}^{(m)} = \sum_{m} \mathbf{D}_m
\]

(cf. Fig. 5A), a matrix which has the correct eigen-directions (principal axes of the tensor-ellipsoid), while the eigenvalues may be uncalibrated (incorrect).

A remarkable feature of the scheme proposed in Figs 4 and 5 is that in the reverberative search for the subsequent second and third eigenvectors, the filtering-out of the already imprinted eigenvectors can be automatically provided by the eigendyads which have already been generated. For example, after having imprinted the first eigendyad, the reverberative search for the second eigenvector requires the \( \mathbf{p} - \mathbf{p}_r = \mathbf{p} - (\mathbf{E}_2 \times \mathbf{p}) \) motor output; the vector which contains only that component of the reverberated vector which is orthogonal to the established eigenvector. The mossy fiber input to the cerebellar cortex would transform, through the corticonuclear network, into its eigenvector projection, since the network is "intransparent" to (yields zero product with) vectors orthogonal to the eigenvector. Thus the nucleofugal output will be exactly the required filtered vector. With the use of such filtering via the already generated eigendyads, the reverberation can proceed to find all the subsequent eigenvectors in one continuous series of stabilizing oscillations, as shown by a computer simulation. When all eigenvectors are found and the respective eigendyads are imprinted into the corticonuclear circuitry, the proprioceptive reverberation automatically becomes superfluous, since following the normalization of the reverberated vector, each corticonuclear eigendyad...
The modification procedure of calibrating the eigenvalues, introduced in Ref. \(^7\), is elaborated here by using a concise summary diagram in Fig. 5B. This diagram is shown for two dimensions (since the motor apparatus of Fig. 1 is confined to a plane) and demonstrates the adjustment of only one eigenvalue. Nevertheless, since the eigenvectors are mutually orthogonal, all eigenvalues (even if more than two) can be altered by a single operation in a parallel motor system. The errors in the performance (produced by a network established with incorrectly calibrated eigenvalues) can be used for its iterative perfection.

The procedure is based on the two-stage character of the metaorganization: (1) the establishing, by a reverberative “revolution”, the eigenvectors of the physical apparatus and thus imprinting the eigen-dyads (determining the principal axes) into a network and then (2) calibrating by a gradual “evolution” the eigenvalues of the functional geometry in order to match the exact values along the principal directions to those of the physical system.

The modification procedure of calibrating the eigenvalues, introduced in Ref. \(^8\), is elaborated here by using a concise summary diagram in Fig. 5B. This diagram is shown for two dimensions (since the motor apparatus of Fig. 1 is confined to a plane) and demonstrates the adjustment of only one eigenvalue. Nevertheless, since the eigenvectors are mutually orthogonal, all eigenvalues (even if more than two) can be altered by a single operation in a parallel manner and since it contains no restriction for dimensionality the procedure is valid for any dimensional motor system.

Assume that a given eigenvector, \(E_m\), of the physical system has been properly established, but it was imprinted earlier. Since the metric can be constructed, in this spectral representation, as the sum of eigendyads with the eigenvalue-coefficients, the difference between the existing and desired eigenvalues can serve to correct the eigenvalues of the existing metric by adding the dyad formed by the climbing fiber vector, \(c\).

The modification is based on the fact that the inner product, \(E_m \times e\), represents the orthogonal projection component of the output vector \(e\) of the network-metric to the eigenvector, \(E_m\). Similarly, \(E_m \times i\), represents the projection to the eigenvector \(E_m\), of the input vector \(i\). Thus the eigenvalue inherent in the existing erroneous contravariant metric network is \(\lambda_m = (E_m \times e)/(E_m \times i)\). Likewise the eigenvalue that is implicit in the contravariant metric of the physical effector mechanism is \(\lambda_m = (E_m \times e)/(E_m \times p)\). If the task is to correct the \(\lambda_m\) in order to become \(\lambda_m\), then the correction of the eigenvalue should be:

\[
\lambda_m = \lambda_{m}' - \lambda_{m} = \frac{(E_m \times e)(E_m \times p)}{(E_m \times e)(E_m \times i)} - \frac{(E_m \times e)(E_m \times i)}{(E_m \times e)(E_m \times p)}
\]

This can be accomplished by adding a modification-matrix \(A\), that is the dyadic product of the climbing fiber correction-vector, \(c\) to the existing network-matrix,\(^7\)

\[
g' = c > < c
\]

\[
c = \sum_m \left[ \frac{(E_m \times e)(E_m \times p)}{(E_m \times e)(E_m \times i)} - \frac{(E_m \times e)(E_m \times i)}{(E_m \times e)(E_m \times p)} \right] \cdot E_m.
\]

The process of calibrating the eigenvalues of the metric (by the above modification algorithm) is illustrated below by a numerical example that uses the cerebellar scheme (Figs 2–5).

Suppose that both eigenvectors shown in section 2.3.1 have been properly established but that the second eigenvalue was erroneously set. Instead of using the correct \(\lambda_2 = 0.209\), assume that an incorrect \(\lambda_2 = 0.300\) exists in the imprinted corticonuclear eigendyad. The erroneous coefficient of the second eigendyad results in a matrix-component:

\[
\begin{bmatrix}
0.613 & -0.120 & -0.472 \\
-0.120 & 0.023 & 0.092 \\
-0.472 & 0.092 & 0.364
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.204 & -0.400 & -1.573 \\
-0.400 & 0.077 & 0.307 \\
-1.573 & 0.307 & 1.213
\end{bmatrix}
\]

\[
\begin{bmatrix}
2.155 & -0.280 & -1.458 \\
-0.280 & 0.204 & 0.430 \\
-1.458 & 0.430 & 1.331
\end{bmatrix}
\]

Let an arbitrary intention vector be \(i = (-100, 100, 100)^T\). Thus, through the error-laden \(g'\), this covariant
vector will be transformed into a contravariant
execution vector $e = (-389.3, 91.7, 321.9)^T$. This execu-
tion vector will result in the physical output of $P$
instead of $G$. Through the covariant metric, implicit
in the physical geometry as expressed in section 2.1.4,
the invariant $P$ will be measured by the performance
vector as

$$
p_i = \begin{bmatrix} 1.000 & 0.906 & 0.799 \\
0.906 & 1.000 & 0.978 \\
0.799 & 0.978 & 1.000 \\
\end{bmatrix} \begin{bmatrix} \lambda = -389.3 \\
91.7 \\
321.9 \\
\end{bmatrix}
$$

Since $E_2 = (-0.783, 0.153, 0.603)^T$, the inner
products required for establishing the climbing fiber
vector $e$ are:

$$
\begin{align*}
E_2 \times e &= 512.9 \\
E_2 \times i &= 153.9 \\
E_2 \times p &= 107.2
\end{align*}
$$

The ratio, representing the eigenvalue inherent in the
physical geometry, therefore, is

$$
512.9/107.2 = 4.785
$$

and the ratio, representing the eigenvalue erroneously
implemented in the network is

$$
513.0/153.9 = 3.333.
$$

From the above, the required correction is $\lambda =
4.785 - 3.333 = 1.452$, and thus the modification-
which, to be imprinted into the corticonuclear
network is

$$
\begin{bmatrix} 0.613 & -0.120 & -0.472 \\
1.452 & -0.120 & 0.023 \\
-0.472 & 0.092 & 0.364 \\
\end{bmatrix}
$$

This modification-dyad added to the erroneous
second dyad of (18) results in the proper second dyad
as shown in section 2.1.4.

The dyadic product of the climbing fiber correction
vector, $e$, can be impressed on the corticonuclear
network as a whole, via CFs that project both to the
PCs and the CB nuclear cells. The emerging cortico-
nuclear matrix will then act as an appropriate metric
producing a zero error in the next performance.
In this trial-and-error process the internal geometry
becomes increasingly homeometric with the external
one. Thus through the CF system the physical geometry is matched with its proper internal
representation.

In the example it was assumed that the first eigen-
dyad was imprinted with the correct eigenvalue.

However, since the eigenvectors are mutually ortho-
gonal the modification procedure is independent along
each eigenvector and thus the modification of all
eigenvalues may be implemented simultaneously in
a single reverberation.

The means of implementation of the proposed
metric-modification at a single neuron level (either at
the PCs or at NCs or at their conjunction) has not yet
been conclusively established. Such a task is all the
more difficult, since one component of the correction-
matrix may either be positive, negative or zero.28

Therefore the required perturbation at a single-
neuron level should be expected as a bimodal effect,
including at times an indetectable zero action.10,17,37

Moreover, the correction in any matrix component is a
function of all vector elements. Thus if only a single
dimension is controlled by the experimental para-
digm, as in conventional analyses, the prediction of
a single component of the matrix may prove to be a
very complex matter. Finally any alteration is ex-
pected to be much more pronounced at the site of the
dyadic convergence (at NC), as opposed to the site of
the intensive search, the CB cortex. These factors,
plus a lack of a conceptual framework accounting for
what is intended in Ref. 78 as the "CB functional triad;
co-ordination, timing and adaptation", may explain
the meager experimental results, despite dedicated
efforts through one and a half decades, in an attempt
to conclusively demonstrate an adaptive feature of
the CB at the PC level.34,46,37

Some additional comments are warranted regard-
ing the operations of the inferior olive. It is assumed
that the inferior olive expresses the difference between
the eigenvalues in the external physical geometry and
those in the functional geometry implemented by the
corticonuclear cerebellar network. The equation
yielding the climbing fiber vector $e$ [introduced in
Ref. 78 and elaborated in (17)], is one of several
possible implementations for the task. The advantage
of the proposal above is that it measures the eigen-
vector of the physical geometry and the eigenvalue
inherent in the already generated network. The dis-
advantage of this solution is that it requires non-
trivial vectoral calculations in the olive, although it is
known that the olive does receive of all the ascending
and descending signals necessary for such a "com-
parator function".7,65

It is therefore noteworthy, that a simplified oper-
ation could also be utilized by the olive, one not
based on measuring the eigenvalue of the cerebellar
neuronal network. Rather, it could utilize the stored
eigenvalue that can be imprinted into the olive at the
same time as the eigendyads are imprinted into the
corticonuclear network. Relying on the stored eigen-
value, its comparison with the error signal can be
used to determine the required modification of the
existing eigenvalue. The error $d = i - p$ may arise as
a difference between intention and proprioception or
between intention and performance, as detected
through the total sensorimotor loop. Thus, the new
eigenvalue should be:

$$\text{new } i_m = \text{old } i_m + (E_m \times d)/(E_m \times p)$$ (27)

This formula, that interprets the climbing fiber vector as based on the error vector, corresponds well to the experimental evidence showing that climbing fibers express functional errors in intrinsic frames of reference.\textsuperscript{61,90} This requires a “computation” in the olive that is simpler than the full formula given in equation (17). The disadvantage of the simple computation is that it may accumulate errors, since instead of measuring the actual eigenvalue of the network the process relies on a stored, and possibly imperfect, eigenvalue by which the network was generated.

A final, but most important comment is that the process of calibrating the eigenvalues by iteration can perfect the metric if the eigenvectors have been precisely established and only the eigenvalues are incorrect. However, if the eigenvectors themselves are improper, the above modification process may continue indefinitely without ever converging to the proper metric. If the intrinsic system of co-ordinates is experimentally altered and the old eigenvectors are entirely improper, the system needs to regress to a revolution (a drastic re-assessment of the principal directions and values) by re-doing the entire oscillatory reverberative process. This prediction of the theory corresponds to the findings that after major disturbance to the cerebellar coordination-apparatus (e.g. vestibular nerve ablation),\textsuperscript{58} one of the earliest and most dramatic phases of the compensation process is marked by violent shaking and oscillatory behavior, observable in the animal and at the olivary level.\textsuperscript{58}

3. GENESIS AND MODIFICATION OF THE THREE-STEP NETWORK OF THE TENSORIAL SENSORIMOTOR SCHEME

3.1. The three-stage tensorial scheme of sensorimotor systems

In section 2, the mathematical principle and the physical process of the metaorganization was elaborated. It was applied to the metric-type motor network that organizes the functional geometry of a neuronal system through the generalized inverse of the covariant metric of a motor apparatus. With this background the three basic transformation matrices used in the tensorial sensorimotor scheme may be developed in principle as well as in physical reality. The building-blocks of the system are (a) a matrix which serves as a contravariant sensory metric, (b) a matrix which expresses the covariant embedding involved in sensorimotor transformation and (c) a matrix which serves as a motor metric. The entire procedure will be demonstrated in the model scheme shown in Fig. 6.

The function of these transformations, once the matrices are available, has been quantitatively demonstrataed elsewhere.\textsuperscript{78} The system shown in Fig. 6 has been simplified and the diagram serves only as a model by showing how the above three matrices may be generated.

3.2 Metaorganization of motor geometry: cerebellar metric-type networks

The first step must be to generate, via the metaorganization process, the neuronal networks comprising the functional geometry of the executor mechanism. In the case shown in Fig. 6(A), this means generating the cerebellar network g.\textsuperscript{44} The motor geometry must be established first since the metaorganization process applies as soon as a motor system and the proprioceptive system are available and connected to the external physical reality. At this stage the process does not require the sensorimotor transformation matrix or the sensory metric. This is in contrast to the generation of the sensory metric which, as will be shown in section 3.4, presumes the availability of the motor metric. Indeed, as pointed out recently\textsuperscript{49} such order in the developing of the particular networks is consistent with the classical morphological studies.\textsuperscript{38} In particular, in the case of the cerebellum, the first part to be developed is the cerebellar nuclei and then the Purkinje cell nuclear pathways, followed by the development of the input to Purkinje cells. After this the connectivity is refined.\textsuperscript{44,88} Accordingly, the metaorganization of motor geometry, as the first stage of development, may start as soon as the muscles and proprioception peripheral organs begin to function. This agrees with the observations of embryonic twitches\textsuperscript{8,30} and the fact that in some species cerebellar neuronal networks are reasonably well developed at the neonatal stage.\textsuperscript{3}

The development of the motor-metric-type cerebellar network is shown by Fig. 6(A), according to the steps described in section 2. There are two additional points to be considered.

The calibration of the proprioceptive vector components can be accomplished by using the ratio of the spike-frequency bursts from the motoneurons which innervate each muscle (shown schematically in Fig. 6B) and the registered response returning via the proprioceptive reverberation. This is possible since the “base vector” represented by a single-motor impulse has only one non-zero component and thus it is therefore both co- and contravariant. For the same reason the calibration is independent of whether the calibrator signal is transmitted through a developed cerebellar metric or entirely bypasses this circuit without undergoing a cerebellar covariant-contravariant transformation.

A second point follows from the fact that tensor analysis deals with general co-ordinates. Thus, while in order to keep the complexity of the presentation minimal, only two-dimensional spatial co-ordinates are shown here, the principles apply to any multi-dimensional system such as one with space- or torque-time etc. co-ordinates. Thus the generation and
modification of metrics involved not only setting or altering the connections, i.e. the electroresponsiveness of the components in the metric network, such as required for altering the spatial metric, but also involves a modification of the dynamic temporal characteristics of neurons. One manner in which this can be accomplished refers to changing the zero-first-second order time-derivative properties of single neurons, in effect changing the characteristic oscillation frequencies of the neurons or of small assemblies. If findings on ongoing modulation of the electroresponsiveness at the PC level are made unambiguous, they could be interpreted as means for such subtle modulation of the Taylor-coefficients of the Purkinje cells in the temporal lookahead-module but not the means of the setting of the eigenvalue-coefficients of the corticonuclear eigendyads.

3.3. Development of the sensorimotor network, implementing the covariant embedding transformation

Although the sensory system shown in Figs 6(B) and (C) is diagrammatic (being composed of two non-orthogonal axes at 150 and 270°) the basic features relevant to tensor theory are well represented; (a) the sensory frame covariantly measures the action generated by the muscles and (b) it utilizes a non-orthogonal system that is different from that for the motor execution both in its direction and number of axes. In the analysis of sensory mechan-
isms such as the six semicircular canals of the vestibular apparatus, both the anatomical realism and the ensuing quantitative complexity is significantly greater. In both the realistic and simplified cases, however, attention must be focused on the transformation matrix that changes the sensory vector into the motor vector (Fig. 6B). The function of the sensorimotor covariant embedding matrix is discussed in detail in Ref. 78. In the case of the vestibulo-ocular reflex model (see Fig. 5 in Ref. 79), the process of establishing such a covariant-embedding matrix was qualitatively elaborated to show that each unit-vector of the motor system should be covariantly measured along the sensory axes, yielding a matrix of the cosines among the sensory and motor axes.

Mathematically, this is a trivial operation both in the simple model presented in this paper and in the more complex vestibulo-ocular reflex. The matrix elements are the cosine-projections of each motor unit-vector onto the sensory axes. Thus, in the case shown, the motor axes with 185, 160 and 148° have to be projected onto each of the sensory axes with 270° and 150° angles:

\[
c_{ij} = \cos (\phi_{ij}) \\
= \begin{vmatrix}
\cos (185° - 270°) & \cos (185° - 150°) \\
\cos (160° - 270°) & \cos (160° - 150°) \\
\cos (148° - 270°) & \cos (148° - 150°)
\end{vmatrix}
\]

\[
= \begin{vmatrix}
0.087 & 0.819 \\
-0.342 & 0.985 \\
-0.530 & 0.999
\end{vmatrix}
\]

(28)

In order to establish these components of the sensorimotor transformation matrix in the CNS, it is necessary to assume that each “base vector” of the motor system is generated by a spike burst of a premotor (Pyramidal-type) neuron, such as are schematically represented by Fig. 1 in Refs 75, 76 and 78 and by Fig. 6B in this paper. These cells are designated here as “premotor neurons” because they use the motor frame of reference and are capable of generating a movement. However, since they will be connected to the sensory mechanism by the covariant-embedding matrix, they express the displacement in a covariant manner. Therefore a direct execution of such signals without a cerebellar-type covariant-contravariant transformation would result in a dysemetic movement. The firing of each large Pyramidal-type premotor neuron can be evoked by a strong burst of activity in a small cortical cell (e.g. layer IV), while Golgi-type inhibitory neurons ensure that other large Pyramidal cells are silenced during this operation (cf. Fig. 1 in Ref. 76). Thus during this special sensorimotor “imprinting” stage only one vertical column of a Pyramidal cell may produce an excitation at one time, producing a “base vector” signal that descends to the motor mechanism and generates a movement. It is noteworthy that these “base vectors”, as in the calibration process proposed in section 3.2, are both co- and contravariants. Thus in establishing the sensorimotor matrix, the cerebellum is again “transparent” as a covariant-contravariant transformer; it is not necessary for the cerebellum to partake in this procedure.

Second, the elementary movements will be measured covariantly by the sensory system. Since the sensorimotor transformation is a pre-requisite to the establishing of the sensory metric, it follows that the sensorimotor transformation must precede the development of the sensory metric. However, direct connections between the input and output elements of the sensory system are necessary and thus it is assumed that in an initial state the matrix of the sensory metric is a Kronecker-delta, i.e. a set of such direct connections. This permits the covariantly measured sensory components of the motor base vector to be transferred directly to the Pyramidal cells. This covariant vector will produce a synaptic activation that yields the exact coefficients to be imprinted into the premotor neuron and thus result in the required components of the sensorimotor matrix (cf. Fig. 6B).

Note that the two procedures of proprioceptive calibration and sensorimotor imprinting can be combined into a single process, since both rely on the “straddling” of the motor system by the unitary firings of individual actuators. However, while the motor-metric network may be organized during embryogenesis (since it only requires motor effectors and proprioceptive reverberation), the development of certain sensory mechanisms (for instance, vision) must commence postnatally.

3.4. Metaorganization of the sensory geometry: tectal metric-type networks

The final stage of the emergence of sensorimotor networks is the development of a sensory metric (Figs 6C and D). This requires not only the availability, but the active participation of the developed sensorimotor embedding transformation and the cerebellar motor metric networks. The development of a sensory metric by means of a motor metric as proposed here is consistent with (a) the presence of a separate, explicit CNS sensory metric that expresses invariants both covariantly and contravariantly in the same frame of reference, the inner product enabling geometrical judgements on the invariant and (b) the fact that sensory functions such as vision cannot emerge without the active participation of motor mechanisms, such as eye movements.

The process is based on a generalization of the metaorganization principle. Metaorganization could develop a network that implements a secondary geometry when the primary geometry was inherent in the motor executor mechanism (see section 2). The physical geometry of the musculoskeletal system provides a proprioceptive covariant vector which is a counterpart for every contravariant motor action.
The generalization of the metaorganization principle is based on the recognition of the fact that any system, not only a physical apparatus, that is capable of providing the dual counterpart to any particular input vector can serve as the primary geometry and thus be duplicated or complemented by the process of metaorganization.

Note that the sensory metric network on one hand and the rest of the sensorimotor scheme on the other constitute two halves of a circle, which are joined through an external invariant. After completion of the development of the sensorimotor embedding and the motor metric networks, however, the undeveloped "sensory metric network" is still only a set of input and output neurons which have the presumed direct system of connections (a Kronecker-delta) from the sensory receptors to the sensory perceptors. Looking at the input and output neurons of this "sensory metric network" reveals, however, that while the input neurons may be improperly connected to the sensory output neurons within the sensory metric network: the output neurons are properly connected to the sensory input neurons through the periphery via the sensorimotor metric networks: the external motor machinery and the sensory apparatus. That is, while one half-circle is still unorganized, the other is perfectly functional. Thus any arbitrary vector, $s'$, over the set of sensory perceptor neurons (even one generated by internal "noise" of the system) can be transformed through the sensorimotor embedding network, $e_s$, into motor intention vector $i_s$, then through the cerebellar motor metric into motor execution vector, $e''$. The physical invariant, emerging from the contravariant motor execution will then be covariantly measured by the sensory mechanism to yield a sensory reception vector, $s$. Therefore, as shown schematically in Fig. 6(D), to any contravariant $s'$ the total pathway that includes the external physical motor and sensory mechanisms will yield the appropriate covariant counterpart, $s$. This is the necessary and sufficient condition for the applicability of the metaorganization process, which then can serve to modify the geometry of the contravariant sensory metric from the Kronecker-delta to the actual functional representation of the remaining circuit.

Given simplified sensory and motor frames, the reverberation process of $s$, (as if it were $s'$) through the entire motor and sensory systems, will yield the normalized eigenvectors $E_1 = (2^{-1/2} \ 2^{-1/2})$ and $E_2 = (2^{-1/2} - 2^{-1/2})$ with the corresponding eigenvalues of $\lambda_1 = 0.5$ and $\lambda_2 = 1.5$. Therefore, the contravariant sensory metric can be generated in its eigendyad-expansion:

$$g'' = \begin{pmatrix} 0.500 & 0.500 \\ 0.500 & 0.500 \end{pmatrix} + \begin{pmatrix} 1/1.5 \\ -0.500 & 0.500 \end{pmatrix} - \begin{pmatrix} 0.666 \\ 1.333 \end{pmatrix} \begin{pmatrix} 1.333 \\ 0.666 \end{pmatrix}. \quad (29)$$

This sensory metric, shown in Fig. 6 as $g''$ (elaborated in Fig. 1 in Ref. 76), can be implemented (using a scaling factor of 4/3) as a simple neuronal network with twice as many direct connections between sensory receptors and sensory perceptor neurons as the number of cross connections.

Since the metaorganization procedure applied here is conceptually identical to that described in section 2, only two general comments will be made here, in order to illustrate how the proposal of establishing the sensory metric fits into the hierarchy of the top-down and bottom-up approaches used in brain theory. At the top level, the process of finding the eigenvectors of the sensory metric by reverberations, would look like a rhythmic oscillatory exercise of the motor mechanism, accompanied by an intense introspective use of the sensory apparatus—similar to the behavior observed during embryogenesis. At the bottom level of the neuronal circuits subserving sensory information processing, the primary covariant-contravariant sensory transformation may take place in the neuronal network such as the optic tectum (cf. elaboration in Ref. 74).

The suggestion that such a network could be generated by generalized metaorganization raises a novel functional interpretation of the actual neuronal circuits that are known to be involved in sensory preprocessing. The process of metaorganization requires that the eigenvectors, found by reverberation, be stored. The implementation of this function can be accomplished by neuronal networks of small nuclei such as the inferior olive rather than cortical networks where the spectral expansion of the metric by its eigendyads is implemented. However, in order to serve as the "imprinter" and continuous "corrector" of the cortical network, as explained by the metaorganization process, such a nucleus must be intimately connected to a cortex. A possibility exists that the nucleus isthmi, known to be endowed by the above-described properties, could play a similar role in generating the tectal circuits.

4. DISCUSSION

It seems quite clear that an analysis of basic sensorimotor transformations requires the use of general hypotheses regarding how the brain might implement functional geometries. The principle of metaorganization is capable of embodying such a concept since it is general enough to encompass many features of related geometries, yet can be elaborated (as in this paper) in specific network models.

A basic assumption of the metaorganization process is that the set of relations among the elements and among elements of another system may exhibit certain common basic features, since one may be embedded into the other. Identity of the two geometries, however, is not required; in the analyzed case the points of one space represent physical locations in a Euclidean space, while the internal functional
motor hyperspace is of higher dimensionality and non-Euclidean (not even Riemannian).

A most practical feature of the metaorganization process is that it enables an explicit study of how an existing and well-defined primary geometry (such as the physical geometry of the motor apparatus), organizes a much less explicit and often ill-defined functional geometry which is implemented by a neuronal network.

The emergence of highly organized structure-functional features of neuronal assemblies is often labeled as "adaptation", "self-organization", or "learning" in the CNS. While the principle of metaorganization and its algorithm is closely related to these notions (which have no precise and generally accepted definition), it differs from them in several fundamental respects. First, the geometrical re-definition of the emergence of networks in metaorganization is based on an identification of two entities: (a) one that governs the process of organization and (b) another that is being formed. For example, the first geometry is defined as that arising from the physical structure of the effector system, while the latter is defined as an abstract geometry over a multi-dimensional manifold. The well-defined nature of these two entities, which mold one another, tempts one to compare them with the notion of "self-organization", which has generated much interest without, however, defining the entity that is responsible for the organizing the "self". Second, since the metaorganization principle is elaborate using formal geometrical analysis, tensor network theory of the CNS and the process of metaorganization can be demonstrated quantitatively by using specific neuronal networks.

4.1. Metaorganization of CNS hyperspaces: a geometrical re-definition of the notions of "adaptation", "self-organization" and "learning"

Adaptation, self-organization and learning have been conceptualized and elaborated using many different approaches.

Viewing the CNS as an adaptive control system represents two aspects of the CNS. "Control" insures that a system conforms to the internal order defined by the neuronal networks. "Adaptation", in turn, enables the CNS to conform with the external conditions. One limitation of an approach that separates, rather than unifies, control and adaptation may be that it assumes the concepts and formalisms of control-system theory. Borrowing from engineering, neuroscientists almost invariably chose its most limited form, the feedback-gain control of a single variable. That description did not lead to a network theory of existing neuronal circuits. Formally, the unselective borrowing from control engineering, even in the form of modern multi-variable control theory may lead to major distortions. For example, since engineers express vectors in convenient Cartesian orthogonal coordinates for man-made systems, some neuroscientists may be led to believe that a vectorial notation that does not distinguish covariant and contravariant expressions may still be adequate to describe biological vector transformations. Admitting to the possibility that nature may have selected other than Cartesian co-ordinate systems and that most natural frames of reference are demonstrably non-orthogonal (cf. vestibulo-collic reflex) leads to the inevitability of using a conceptual and mathematical apparatus that can express physical invariants in general non-orthogonal coordinate systems (such as tensor analysis, or other mathematical apparatus as listed in Ref. 75).

In order to compare the metaorganization principle with the hypothesis that the CNS is a self-organizing system, it is necessary to briefly assess the development of the latter concept. The view of the CNS as a self-organizing system originates from automata theory. Basic considerations on the functional organization of the brain immediately elevated self-organization to one of the most intriguing chapters of Cybernetics. Postulating a synaptic mechanism which may underlie the organization of behavior provided a link between the abstract theory of the emergence of neuronal systems and experimental neuroscience. However, research at that time did not provide either a formal definition or a rigorous elaboration of the general notion.

A second major increase in interest in self-organization occurred in the 1960s when self-organization was tied to concepts of "learning", "optimization", "adaptation" and "approaching a steady-state in relating to external systems". It was explicitly stated, however, that a precise definition was still lacking at that time.

Finally, self-organization has attracted intensive theoretical interest again in the last decade. Although the mathematical sophistication inherent in these new models is unsurpassed, a generally accepted definition of self-organization, both in a philosophical or mathematical sense, remains elusive. Therefore the term is used in neurobiology in a largely intuitive sense. It may be applied to a specific phenomenon such as the emergence of temporally stable neuronal patterns, or to such abstract phenomena as human learning. Philosophically, however, the usage of the term "self" to an organizing principle or a process appears contradictory, since the organization of the "self" must surely be separable from the steps which generated it. Indeed, the question may not be, how does the CNS organize itself, but rather, how is the "self" organized in the CNS by the rest of the body and by the external world.

A geometrical redefinition may become helpful in alleviating some of the problems. For example, by providing separate definition for the two interacting geometries that is the geometry of physical features of motor systems and the functional geometry of neuronal networks that co-ordinate their actions, the
causative relationship between these two becomes explicit and quantifiable.

4.2. Oscillations and tremor

One aim of tensor network theory is to serve as a help with a general interpretation of experimental data relating to brain function. It is therefore significant to determine if some basic experimental observations may be formally related to general principles of CNS organization. Indeed, neuroscientists have long wondered if phenomenology as basic as biological oscillation and resonance may provide a key to the mechanism which determines the organization of the neuronal networks of the CNS. This possibility is supported by the argument that oscillations and resonance are observed during development at several levels of the neuraxis. Such processes may thus be the means of development of electrical properties of the neuronal elements that constitute the neuronal nets. In fact, neuronal oscillation and resonance determine much of the developing limbs and thus must provide, by recurrent afferent activity, crucial information about the dynamics of body-reference frames during early neurogenesis. Moreover, it is quite possible that such an internal searching mechanism may be operant in the adult form and may become quite explicit in pathological conditions. For example, it has long been known that in patients with Parkinson’s disease, mechanical oscillatory stimulation of a finger may induce tremor which irradiates upwards along the limb, in a manner similar to the Jacksonian “march” of motor seizures following localized lesion of the motor cortex. This “tremor–march” phenomenon indicates that the several segmental levels of the CNS which control limb movement are coupled to each other such that they may phase-lock and resonate when tremor occurs in only one segment. Such dissipative functional structures are of the essence when considering that the entire limb may be used as a single element or as a set of separate compartments. The importance of the interaction of sensory feedback and motor output becomes clearer when considering that this interaction begins to occur very early in development even prior to the generation of co-ordinated movement and could thus serve as an epigenetic organization influence in determining the selective stabilization of neuronal networks.

4.3. Generalization of the metaorganization principle: tensorial interpretation of the hierarchy of dual (complementer and duplicate) geometries in the CNS

The principle of metaorganization was elaborated in section 2 for a motor system where the primary geometry was directly identifiable. Active muscular forces are thought to represent contravariants, while passive tendon-forces represent covariant expressions of motor action. This primary physical contravariant–covariant transformation was complemented by establishing a secondary functional geometry which implemented a covariant–contravariant operator. This was accomplished by finding the eigenvectors and eigenvalues of the existing motor mechanism. Several generalizations of the metaorganization principle lead to its application to systems that are beyond such primary geometries as those inherent in motor mechanisms.

The first generalization of this principle was made possible by the realization that the primary geometry does not have to be represented by a single physical metric transformation, but could be an entire sensorimotor chain which becomes a completed circle through its interaction with an external physical invariant. This was the case when constructing the sensory metric (Fig. 6D), where the entire sensorimotor chain provided for any contravariant sensory perception vector, \( s \), the corresponding covariant perception-vector, \( s^\top \). This was attained through (a) a sensorimotor neuronal network, (b) a motor metric operator and (c) a physical invariant generated by the motor system where this invariant was physically measured by the sensory apparatus.

Beyond the above, the metaorganization principle may be applied to any metric tensor (e.g. one that is manifested only in an abstract functional geometry). For instance, once the network of the sensory metric is developed as a geometry which is secondary to the primary geometry inherent in the external world, the sensory metric itself may then be used as a primary geometry for metaorganizing its own secondary geometry in a hierarchically coupled neuronal network. For instance, the optic tectum itself may generate a secondary space in the cerebral cortex (that is "empty", i.e. "amorphous" in its pristine state), by serving as the organizer of the higher-order mold in this cortex of the tectal primary functional geometry.

The third generalization follows from the simple fact that the metaorganization process builds the secondary geometry from the eigenvectors and eigenvalues of the primary geometry. In generating the eigendyads, metaorganization could utilize either the eigenvalues of the initial system themselves of the initial system or their generalized inverses (or both). Metaorganization can thus result either in a secondary geometry which is a duplicate or a complement of the primary geometry, or both. This generalization leads to a study of hierarchically connected hyperspaces where the primary spaces (comprising sensorimotor metrics) are directly connected to physical geometries external to the CNS but where the secondary hyperspaces are built upon these sensorimotor metrics. Each geometry may in turn breed higher order duplicate or complement geometries in connected hyperspaces. As pointed out, the intelligence implemented by such a hierarchy of metageometries may depend on (a) the number of levels of connected
of a sensorimotor mechanism could be established if

4.3.1. Biological generalization of metaorganization: nucleocortical structures as metagenerators. While the theoretical implications of the above proposals may be far-reaching, the immediate task is to systematically close the gap between experimental data and theory. This must be accomplished at the system, network, cellular and subcellular levels. Techniques of modeling neurons and neuronal networks\(^\text{[77]}\) may be used to determine if network and system-level proposals are compatible with our knowledge of the physiology and biophysics of single elements. Initial checks of general theories as described above should actually be suggested by the proponents of such theories. First, the relation of nucleocortical CNS systems can be studied to determine whether the nuclear structures serve to embody the functional eigenvectors and eigenvalues of primary geometries or whether the cerebellar cortical structures embody eigendyadic expansions implementing the secondary functional geometry as determined by the eigenvectors and eigenvalues. A particularly relevant possibility is that the thalamocortical and other similarly arranged neuronal networks could in fact be dynamically organized as proposed here for the olicocerebellar neuronal apparatus.

Second, because of its possible use in system-level research the limitations of the metaorganization principle also need to be pointed out by the proponents. As shown in section 3, the three neuronal networks of a sensorimotor mechanism could be established if the system was organized in a closed loop with regards to one physical invariant (in the example in section 3 the invariant was a physical displacement). This is, however, not the case for systems such as the vestibulo-ocular reflex.\(^\text{[79]}\) There, although the motor metric can be organized by the metaorganization process, the sensorimotor transformation network and the last step, the sensory metric, cannot be similarly generated. Indeed while the oculomotor metric could in principle be generated by metaorganization,\(^\text{[79]}\) since the extracocular muscle activity and its derived proprioception do represent the same invariant eye movement,\(^\text{[80]}\) the CNS cannot use metaorganization to generate the full vestibulo-ocular reflex network, because eye movements cannot be detected by the vestibular apparatus. However, systems such as the vestibulo-ocular reflex can emerge from a hierarchy of primary sensorimotor systems, such as the vestibulo-collic reflex and retino-extraocular reflex.\(^\text{[79]}\) Finally metaorganization can be used, as will be shown elsewhere, to generate the three sensorimotor networks in each primary system and a seventh network to tie the two closed reverberative primary mechanisms into a hierarchical sensorimotor architecture of the vestibulo-ocular reflex. Thus while the metaorganization principle and algorithm is applicable to quantitatively feature the development and function of some specific neuronal networks such as the cerebellum, its ultimate use at other levels of CNS function remains largely unexplored at present.

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Note added in proof:
After this work was accepted for publication, a paper appeared offering some remarks on the uses of tensor network theory [Arbib M. and Amari S. I. (1985) J. theor. Biol. 112, 123-155]. While generally supportive and appreciative of the tensor approach and advocating its further use, the paper also raised five specific points of concern. These can be readily dealt with here as they are off the mark regarding the basic principles of tensor network theory and probably originate from a misunderstanding of our approach.

(1) "Modern mathematics has developed many techniques for coordinate-free analysis of structure. PL [Pellionisz and Llinás] write as if the use of tensor analysis were the only such technique". This remark is incorrect. The tensor approach was explicitly compared to other mathematical techniques, see chapter in Ref. 75 entitled "Comparison of tensor approach with other geometric theories: Representation, modeling, mapping, differential geometry, lie algebra".

(2) "PL speak of a CNS hyperspace F but never prove that F is a Riemannian manifold." This remark is based on a fundamental misunderstanding of tensor network theory. Indeed, engineering typically uses Cartesian tensors of Euclidean space; relativity uses tensors in Riemannian space. What we face, however, is not a duty to conform with previous scientific approaches but a challenge to create methods that conform with CNS function, and thus we have proposed tensor analysis of brain hyperspaces. The important point to be understood here is that CNS hyperspaces need not be confined either to Euclidean or Riemannian geometry. The remark is also incomplete. The non-Euclidean character of the geometry has been demonstrated by a quantitative computer model; see section entitled "Position-dependent metric" of Fig. 5 in Ref. 81.

(3) In tensor network theory "every array of numbers must be the co-ordinates of either a covariant or contravariant vector. . . ." This is not so, for instance, the cerebellar nucleofugal vector [(i-e) in Fig. 2. of the above "Metaorganization" paper, or in Ref. 76] is neither co- nor contravariant. More to the point, we wrote in Ref. 84: the "... task is to establish whether the way of assigning the components to the invariant is a covariant or contravariant procedure". Once mathematical vectors have been assigned by a co- or contravariant method to an invariant, such arrays of numbers can, of course, be manipulated, e.g. by subtraction, so that the primary character of the original arrays is no longer evident.

(4) Tensor network theory "makes no use of tensor theory beyond the metaphorical use of the terms 'covariant', 'contravariant' and 'metric tensor'...". This remark is incorrect. We have used multidimensional tensorial analyses of spacetime manifolds, tensor transformations by covariant embedding, quantitative exposition of the curved character of CNS hyperspaces, covariant-contravariant transformations by the Moore–Penrose-generalized inverse, contraction of tensors and network elaborations. Readers acquainted with some of our other twenty or so papers published on this subject (not cited by these authors) will be aware of these uses and developments.

hyperspaces or (b) the precision with which one metageometry is molded by another.
Metaorganization of functional geometries in the CNS

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