BRAIN THEORY: CONNECTING NEUROBIOLOGY TO ROBOTICS.
TENSOR ANALYSIS: UTILIZING INTRINSIC COORDINATES
TO DESCRIBE, UNDERSTAND AND ENGINEER
FUNCTIONAL GEOMETRIES OF INTELLIGENT ORGANISMS

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Abstract
To facilitate a "knowledge-understanding-utilization" transition from
neurobiology to socioeconomically gainful applications, a conceptually and
formally unified brain theory is required. A perspective is given on the variety
of concepts and formalisms available to brain theorists and engineers, with
special regard to the vector-matrix approach to the parallel system of the brain.
A change in the axioms of the vector-matrix approach has led to a suitable
unified approach: "brain vectors" ought to be regarded as expressions of
invariants in the natural non-orthogonal coordinate systems of the brain rather
than Cartesian vectorial expressions. Some results of the tensor network theory
of the brain are listed. Immediate possibilities for the further development of
this geometrical theory are shown — mostly relating to the question of how
geometries may organize one another in the brain hyperspaces. It is indicated
how theory may facilitate our understanding of brain function and thus catalyze
the development of new types of geometrical brain-like robots.

1. INTRODUCTION

Theories, maps, geometries and other formal expressions of systems of relationships
can be rated according to their intrinsic qualities. Such measures are, e.g., their
self-consistency and contiguity; i.e. the absence of contradictory or missing points, or
the size of their claimed domain of relevance; i.e. whether they describe local and/or
global relationships. The proof of their extrinsic quality, the match of the existing
system of relations and their formal expressions, however, is in their actual utilization.
Such tests of their usefulness occur when a formal expression of the
system of relationships is used as guidance for interacting with, or attempting to
duplicate, an existing system. For instance, a comparison of the calculated and
measured geometrical distance from points A to B, or navigation from here to there
according to a chart, or the attempt to make a model of the path, provide such tests. In
brain theory such tests occur, e.g. in explaining, in a hardware-implementable
manner, how the cerebellar neuronal network makes the coordinated movement of a
multi-jointed limb from point A to B possible.

As a whole industry is presently in an explosive stage of development, scrambling
for utilizable understanding of how intelligent (i.e. brain-like) systems can be
constructed, it is not irrelevant for either theoretical neurobiology or robotics to
assess how brain theory fares in passing such tests of usefulness.
1.1 The Challenge: To Connect Neurobiology to Robotics

Robotics may change the neurosciences as much as nuclear technology changes physics. It is likely that the social impact of the transition will be comparable. These are widely shared impressions, especially among those who closely monitor another similar upheaval in molecular biology under the pressures of genetic engineering. While robotics has already been corroborated with brain research, formal requirements for their merging were not satisfied (Arbib, 1972, Albus, 1979).

The fundamental fact is, that whether dealing with the atom, the double helix or the brain, after a certain point understanding is no longer a goal in itself, but it may become the basis of satisfying the needs of a larger community through a new industry. The fusion of neuroscience with robotics may make possible the implementation of tiresome tasks by man-made devices which would alleviate extreme strain to both the human body and intellect and which would evolve competitively to outperform their peers and creators.

Neuroscience represents what is thus far known about existing intelligent systems. Robotics will eventually aim at making use of whatever is to be learnt from Nature’s solutions. Thus the potential for a “knowledge-understanding-utilization” transition from neuroscience to robotics already exists. A question is whether the scientists and engineers to be involved are ready to institutionalize the impending transition. It can be argued that presently the evolution of robotics is independent of neuroscience, yet mighty organisms can still be produced. While most of these creatures are recipients of a massive infusion of computer-rigor, they possess limited versatility and intelligence — they are comparable perhaps to Nature’s dinosaurs. The evolution is to be won not by body weight, muscle strength or even precision; the primates of robots will be those which elegantly outsmart their evolutionary competitors. The race is on, to create robots with superior intelligence.

1.2 The Requisite: A Conceptually and Formally Unified Brain Theory

An actual realization of this potential puts neuroscientists to a double test. First, the proof of the theoretical understanding they claim is in the application, i.e. whether it provides the blueprint for successful implementation. Second, and equally as important, the use in engineering of the understanding of brain function demands that a unified conceptual framework with a matching common language be developed.

Of course, a conceptually unified and formally homogeneous theory in itself is a perpetual goal of science, if for nothing but the aesthetic pleasure of doing away with discontinuities between separate theories. However, when it comes to merging fields as seemingly disparate as neuroscience and robotics, a homogeneous treatment is no longer merely an aesthetic goal, but a practical necessity. If the neuroscientist who acquires the knowledge of existing intelligent systems of the brain cannot speak the same language as the engineer who is to design their counterparts, fusion is not possible.

Brain theory and robotics

The merging is effected on two levels: formal and conceptual. Regarding such syntheses of burgeoning experimental approaches in neuroscience, the following views are typical. “There has been a tremendous explosion of data over the past decade or so but there is a surprising absence of theory to bring it all together” (Lewin, 1982), and “what is conspicuously lacking is a broad framework of ideas within which to interpret all these different approaches. . . It is not that most neurobiologists do not have some general concept of what is going on. The trouble is that the concept is not precisely formulated” (Crick, 1979). While the need for unifying theories is recognized, attempts may be hindered if an abstract (mathematical) formalism is lacking (Edelman and Mountcastle, 1978, Eccles, 1981).

The needed medium for formal and conceptual merging must a) be based on general concept of brain function, b) be applicable to the available description of its functioning; and c) lead us to an understanding which is mathematically stated by a formalism fitting the basic concept. Finally, since a utilization must occur through the hands of engineers, d) it must be a language accessible to both neuroscientists and the engineering community.

An approach has been developed in recent years that we feel brings us closer to satisfying these requirements (Pellionisz and Llinás, 1978, 1979a,b, 1980a,b, 1981, 1982a,b). To show how this contribution fits to the existing approaches, a perspective is given below on the choice of concepts and formal methods (section 2). This is followed, in section (3), by an overview of some notions of the tensor network theory that are already established in core-papers, and those which are yet to be mathematically formulated.

2. THE CHOICE OF CONCEPTUAL AND FORMAL APPROACHES TO THE BRAIN

It appears that despite the wide variety of functional characterizations of the brain, each can be traced to one, or a combination of several, of the relatively few general concepts. While an attempt to sum-up and size-up these basic ideas in the scope of this paper risks the appearance of oversimplification, it seems impossible to avoid such distillation of the underlying fundamental assumptions, if one is to assess their relevance to both abstract understanding and utilization and if one has to make a choice of which is most suitable for the required knowledge-understanding-utilization transition.

2.1 The Choice of Basic Concepts of Brain Function

The first modern attempt to condense the essence of brain function into a single quantitatively expressive idea led to the emergence of cybernetics. Coined and defined by Wiener (1948) as “Control and Communication in the Animal and the Machine”, it conceptualized the brain around two ideas. This, in itself, may indicate more of a choice than a unification. Moreover, as the reminder below points out, this dual concept, albeit revolutionary, seems to leave out certain significant features of CNS function.
The Brain as a Control Device — Linear System Analysis and Control Engineering.

In the “control system” of the vestibulo-ocular “reflex” (VOR), the brain forces the eye into the opposite direction to which the head turns, in order to stabilize the retinal image. This sensorimotor system, indeed, became the supreme biological model to which control theory was extensively applied (cf. Robinson, 1968). Although that theory was developed for man-made, spatially concentrated, serially organized electronic systems, brain research, especially of the VOR, gradually became thoroughly infiltrated with ideas and techniques transposed from this field of engineering. This generated a great impetus in quantitative analysis of brain function (see Robinson, 1968). Still, to conceptualize the brain as controlling a single variable may not do justice to the complexity of the CNS, as has been stressed repeatedly (Pellionisz and Llinás, 1979, 1980a, 1982a,b).

Another feature that poses a challenge to conventional feedback-control appears to be the variability of such control. Not even the vestibulo-ocular “reflex” is limited to forcing the eye into the perceived direction of a head movement if this type of control is not working: “A final problem ... is the question of plasticity or the ability of the system to shift its reference in accordance with the environmental requirements” (Young, 1969). Indeed, direction-reversing prisms will cause the VOR to “learn”, i.e., to reverse the “gain” of this “control reflex loop” (Gonshor and Melvill-Jones, 1973). Evidently, there is more to CNS function, even for VOR, than control; the brain is as much for learning as for control. More complex functions such as coordination, association, pattern recognition, vision, etc. are even less readily covered by the terms of control.

The Brain as a Communication Device — Information Processing and Computers.

At the time cybernetics was gaining ground, the second part of the dual conceptualization centered around “communication”. From the viewpoint of the contemporary information theory (Shannon, 1948) the brain appeared remarkably similar to an information transmitting and processing device. This impression was reinforced by the analogy between binary electronic elements and the all-or-none character of nerve cell firing. The brain appeared, both in terms of “hardware elements” and implied mathematical concepts (Boolean algebra; McCulloch and Pitts, 1943), to be akin to the computer until the differences were clearly stated by von Neumann (1958).

The Brain as a Learning Device — Artificial Intelligence and Perceptrons.

While the concepts of control and communication led to explanations of the function of a limited number of neuronal organisms, computer science and electronic information processing yielded a bounty of practical (but perhaps not brain-like) solutions to complex “intellectual tasks” — chiefly logical and numerical calculus and the storage of information that became known as “memory”. Success and suggestive terminology triggered expectations that human memory, and — what seemed to be intimately connected to it — learning are within the reach of the abstract sciences. Progress was institutionalized in the form of artificial intelligence research, bypassing neuroscience because it was perceived as not being able to provide an immediately practical base. This led to the conception of the brain as a “learning device”, in the sense of artificial intelligence, as defined by the Perceptron (Rosenblatt, 1962, Minsky and Papert, 1978).

The Brain as a Learning & Control Device — Adaptive Control Engineering.

Since the CNS exhibits features other than those of the perceptron-device, an increasingly favored paradigm is a combination of concepts which describes the brain as both a control device and a learning machine (cf. Ito, 1970). A difficulty of combining these approaches is that “control” is very well defined in engineering, while the commonly used term “learning” is not necessarily so. “Learning” may be meant either ambiguously enough to encompass the addressed neurobiological phenomena, but therefore it is difficult to formally treat it, or meant in strictly defined technical terms — but then the biological feature to be represented may be left out. For example, there appears a gap between learning defined as the process of “associative storage and recall” (in artificial intelligence) and the other “learning”, the reversal of the “gain” in the VOR.

The Brain as a Self-Organizing System.

This concept was born as an attempt to develop a wide basis for conceptualizing brain function by incorporating the ideas of learning and adaptation into that of the emergence of neuronal systems in the form of the Turing-machine (1948), cf. also Shannon and McCarthy (1956), Farley and Clark (1954), Ashby (1962), von Foerster and Zopf (1962). The feature that some systems exhibit a certain underlying capacity of self-duplication, self-generation, self-development, self-modification, self-optimization has attracted great attention (von Malsburg, 1973, Amari, 1982). The difference of perceiving the brain as an adaptive system (where the emphasis is on the influence of an external system on shaping the CNS function) and that of self-organization (where autonomous emergent features of the CNS are stressed) poses the challenge of whether the two can be conceptually unified.

The Brain as a Computational Device. Awaiting a choice of a better concept, the idea of the brain being a computer is still used (cf. Granit, 1981). However, “computational theory of the brain” has been rapidly replacing it in recent years. This view does not identify the brain with a computer either in basic hardware or in underlying mathematical theory. It does assume, however, that the brain performs computations according to some algorithm when it executes a task, and further, it equates brain theory with finding such algorithms.

The so-called top-down approach perceives the chief question in neurobiology as that of finding the kind of algorithm that is capable of performing the task, since “in the neurosciences the top level is a crucial one” (Reichardt and Poggio, 1981). While the truth of this statement is undeniable, it is not insignificant if the top is cemented to the bottom, or the “different levels of understanding are rather independent” (ibid). The argument that “an algorithm is likely to be understood more readily by understanding the nature of the problem being solved than by examining the mechanism ...” (Marr, 1982) has been the basis of finding a computational algorithm, e.g. for vision (ibid) or visuo-motor control (Reichardt and Poggio, 1981, Bizzi, 1981, Poggio and Reichardt, 1981).

A difficulty in the computational theories is that the answers to the two questions...
“How can one perform tasks executed by the brain?” versus “How does the brain perform them?” are differently motivated, and may be quite different from one another. The difference in the emphasis can be dealt with by putting different weights on the goals: 1) by aiming somewhat more at the mathematical perfection of the description of function rather than on its biological relevance, or 2) by aiming explicitly at mathematical descriptions of phenomena rather than their explanation. Such genuine mathematical approaches vary from the most profound mathematical analyses (Caianiello, 1968, Amari, 1982, Grossberg, 1982) to explicitly phenomenological models (Hodgkin and Huxley, 1952).

While computational approaches often excel in the cohesion of the applied mathematical formalism, they may sometimes be more eclectic in the utilized underlying concepts. Indeed, algorithms based on any concept may belong to this group, e.g. systems analysis models in computational clothing (Reichardt and Poggio, 1981) or the so-called “look-up table” paradigm of motor control (Raitbert, 1977) — an approach which conceives movements as the recall of a stored sequence of movement-elements. The challenge inherent in the computational approach appears to be how to synthesize the top-down and bottom-up approaches, since only their intimate combination is likely to succeed.

2.2 The Choice of Abstract Characterization in the Absence of a General Conceptual Framework: Neuronal Modeling

Rather than forging a solid body of science, cybernetics led to some fragmentation of the information and brain sciences. The first major international congresses already demonstrated a sizable, but disintegrating body (Proctor, 1969 Rose, 1970). This was not only from a conceptual point of view, but maybe even more so in relation to applied mathematical techniques, cf. Zadeh (1969) listing 18 methods. The fundamental reason for this dispersion may have been the lack of a coherent conceptual core, for cybernetics did not offer one general concept matched with one formalism. As a result, there is a now a wide gap between “information sciences” (computer science, artificial intelligence) and “brain sciences” (experimental neuroscience and neuronal modeling).

The first group that fell into the realm of “exact” sciences fared quite well, since each branch of the “information sciences” was cohesively held together by the inner logic and formalism of engineering and mathematics. The disintegration of cybernetics, however, had a negative effect on the “brain sciences” by leaving them without central conceptual guidance and generally usable formalism. Instead, the brain sciences split into experimental neuroscience and to neuronal modeling, which dealt with mathematically addressable problems of neuroscience on an ad hoc basis. This era from the early sixties through the seventies was marked by burgeoning experimental approaches. For instance, in the case of the cerebellum, a bounty of knowledge was gathered and it was, e.g. morphologically, evident that the system was a multi-dimensional parallel processor (Eccles et al., 1967). Nevertheless, when attempting a synthesis, there was no choice but to turn to single-variable control theory (Eccles, 1969).

Despite the lack of utilizing a general concept, neuronal modeling proved to be useful (for review, see Jack et al., 1975, McGregor and Lewis, 1977, or Pellionisz, 1979b) — since different experimental techniques have all reached such levels of complexity that a unification of the body of details could only be achieved through the power of pure mathematics or computer calculations (cf. Rall, 1964, Cooley and Dodge, 1966, Calvin, 1969, Jack et al., 1975, Holden and Yoda, 1981) or multi-compartamental modeling of spatially distributed neurons (Pellionisz and Llinás, 1977, Traub and Llinás, 1977, Perkel et al., 1981).

Thus, if an engineer today would have to choose from concepts on brain function (for a more complete choice than it is possible to present here, see the reviews by Reichardt and Poggio (1981), Szentagothai and Arbib (1975), Szekely et al. (1981), Hinton and Anderson (1981)), the question seems open whether one would actually settle for a single one, or a combination of several, or do without assuming a general conceptual framework at all. Yet the need is pressing, since a both conceptually and formally eclectic brain theory makes the knowledge-understanding-utilization transition fragmented, and consequently very difficult.

2.3 The Choice of Basic Formalisms of Brain Function

The parallel nature of the functioning of the brain has been stressed by authors too numerous to mention. Thus, formal methods that are not geared to deal with parallel systems are omitted in this brief overview.

Describing Brain Activity — Patterns, Cell Assemblies, Synergies. That the external world is reflected within the brain in the form of flickering patterns of brain cell activities has been a picturesque analogy since Sherrington’s (1906) “enchanted loom”. Metaphors, however, are no substitutes for formal description when it comes to engineering. One possibility is the analysis of the rather unwieldy mathematical objects of patterns (cf. Beurler, 1962, Katchalsky et al., 1974, Freeman, 1975). Other possibilities are to consider cell assemblies as units of brain activity activity (Hebb, 1949, Legény, 1968, Palm, 1982), or to think in terms of synergies (Bernstein, 1947).

Modeling Brain Activity — Computer Simulation of Patterns. A phenomenological but quantitative handling of “patterns of activities” is possible by a computer simulation approach. This could be used for either arbitrarily constructed neuronal populations (Farley and Clark, 1961) or for realistic (cerebellar) neuronal networks (Pellionisz, 1970, Pellionisz and Szentagothai, 1973, 1974, Pellionisz et al., 1977). Phenomenology, however, must make room for abstraction.

Abstracting Brain Activity — Vectors and Matrices. As pointed out in an overview of the vector-matrix approach (Pellionisz and Llinás, 1982b) it may appear trivial that activity levels of $n$ neurons are often represented by an $n$-dimensional vector (an ordered set of $n$ quantities). Likewise, the set of connections from $n$ input to $m$ output neurons are often described formally by an $n \times m$ matrix. Such a vector-matrix approach, used by numerous authors, is a technically straightforward and powerful formalism, yet not free of conceptual controversy. Pellionisz and Llinás (1982b) surveyed the vector-matrix approach in neurobiology — its introduction (Pitts and
McCulloch, 1947; Wiener, 1948) and some aspects of its use in cerebellar research (Steinbuch, 1961; Szentágothai, 1968). The main fields where this approach is used were indicated — perceptron studies (Rosenblatt, 1962; Minsky and Papert, 1978), theories of associative learning (Kohonen, 1967; von Foerster, 1967; Longuet-Higgins, 1968; Anderson et al., 1972) and “perceptron-type” vectorial control and learning theory of the cerebellum (Albus, 1979).

The vector-matrix approach, while formally eminently homogeneous, is eclectic in the concepts underlying the formalism. For instance, Boolean algebraic and computational theories may be expressed vectorially (von Foerster, 1967), as well as the learning machine of Perceptron (Rosenblatt, 1962, Minsky and Papert, 1978), or the cerebellar adaptive control device (Albus, 1979) or the associative learning theories (Kohonen, 1967).

Despite the standard and ubiquitous usage of the vector-matrix approach, there seems to be a chance to improve on it, by tying it to a general underlying concept (multidimensional geometry of brain function) and by further generalizing the abstract formalism (from Cartesian vector analysis to tensor analysis of general coordinates).

3. THE ATTEMPT TO FORGE A CONCEPTUALLY AND FORMALLY HOMOGENEOUS (TENSORIAL) APPROACH TO BRAIN FUNCTION

3.1 Features Essential to Utilization: Connecting Abstraction with Reality — Mathematical Vectors Assigned to Physical Objects are Treated as Tensors

An improved approach may be built on the general assumption that ordered sets of nerve cell activities (mathematical vectors) are expressions, in the natural non-orthogonal frames of reference used by the brain, of physical objects (invariants) of the external world. That is, brain vectors are now viewed as being expressed in general — as opposed to orthogonal coordinate systems. The original suggestion that neuronal networks could be considered tensors (Pellionisz and Llinás, 1978) carries the same idea since tensors are mathematical entities expressing invariants in any frame of reference, in a generalized vectorial manner.

This article is not an introduction of tensor theory to neuroscience and not the core-paper where primary biological examples are worked out. Such contributions have been offered elsewhere (Pellionisz and Llinás, 1978, 1979a, 1980a, 1982a). This paper attempts to provide a concise characterization of the tensorial approach compared with the alternatives, in order to facilitate a choice for those who wish to develop a dialogue between theoretical neurobiology and robotics. Together with references to the original publications, a simplified quantitative example in Fig. 1. is used throughout the rest of this paper to concisely present basic features of the tensorial approach. Designations are used in the text to refer to parts of Fig. 1, e.g. (d–e 3–4) refers to the $r_1$ block.

At one extreme, the tensor approach could be considered a “trivial” improvement promoting the use of generalized vectorial formalism for the interpretation of many components of neuronal activity as a vector expressed in the brain’s own not necessarily orthogonal frames of reference. However, there is a somewhat overlooked technical distinction that directly follows from the use of general coordinates. In non-orthogonal frames one must distinguish between covariant and contravariant vectors; expressed by orthogonal projection versus parallelogram components (cf. $r_1$ and $p_1$ in (de–36)); since covariant and contravariant expressions are identical only in orthogonal frames. With this fact recognized, the next step is to deal with their tensorial relationships, which is expressed by the mathematical device called the metric tensor. The introduction of the use of covariant and contravariant vectors as tensors into neurobiology in general and oculomotor research in particular is found in Pellionisz and Llinás (1979, 1980a,b). The different mathematical character and physical-biological usefulness of these vectorial versions is of cardinal importance in
tensor theory and must be clearly understood for an assessment of the merits of this approach. These features are shown in detail in Fig. 3 in Pellionisz and Llinás (1980a).

One difference in the properties of covariant and contravariant vectors (as can be seen also in Fig. 1.) is in their method of establishment. The orthogonal projection components of the covariant expressions are established independently from one another — as the sensory components in biological systems. In turn, the parallelogram components of the contravariants are established interdependently, cf. (d–e 3–4) in Fig. 1. The other major difference is in their capabilities. The contravariant components physically assemble the invariant — as motor components in biological systems, cf. (f–g 3–4), while covariants do not. The application of the concept and formalism of covariants-contravariants to sensory-motor systems can be found in Pellionisz and Llinás (1982a).

The basic change in the vantage point of tensor theory, as compared to vector analysis, is the treatment of vectors not only as mathematical abstractions, but as mathematical descriptions in relation to that invariant of the external world to which the vectors are assigned (see de2). This is reflected also in the terminology of invariants, covariants and contravariants (d–f 1–4) instead of referring to all of them indiscriminately as vectors (cf. Pellionisz and Llinás, 1982b). Since the coordinate system defines the relationship of an invariant to its vectorial components, tensor theory shifts the focus from the mathematical vector itself to the properties of the frame of reference and to the geometrical properties of the abstract hyperspace in which the mathematical vector represents a point.

Although the difference in the axioms interpreting brain activity by general vectorial expressions appears to be minor, its conceptual implications are far-reaching. The shift from Cartesian- to general coordinates reverses the direction of the reasoning. The question becomes not how to apply the known mathematics of Cartesian vectors to description of brain function but how to reveal the unknown mathematical features intrinsic to their natural vectorial expression.

The specific application of a tensor-approach to the brain thus needs to proceed by taking the following steps (Pellionisz and Llinás, 1982a,b). 1) Identify the invariant to which a given vector is assigned within the CNS. 2) Reveal the natural coordinate system by which such “mathematical vector” arises in the brain (Simpson et al., 1981). 3) Determine the covariant or contravariant version of the vectorial expression. 4) Establish the geometrical properties of the multidimensional space in which the mathematical vector represents a point. 5) Unfold the relationship of the geometry of this particular hyperspace to the other external and internal geometries.

3.2 A Concise Quantitative Example of the Tensorial Approach to Sensorimotor Systems

Several aspects of tensor theory have already been elaborated in biological examples (Pellionisz and Llinás, 1979a, 1980a, 1982a). Nevertheless, Fig. 1, provides here an additional, quantitative scheme in which some advantages of the approach can be examined (for a detailed elaboration of Fig. 1 see Pellionisz 1984a). The scheme resolves several difficulties encountered when explaining sensorimotor transformation, i.e. the control of movements in systems with a larger than necessary number of executors. This “sensorimotor model” may serve as a symbolic example not just for biological systems, but, e.g., for a robot apparatus — thus the suggestions offered by the paradigm may be suitable for immediate application. The problems, for which solutions are offered are: how does a sensorimotor system accomplish 1) a transformation from one frame of reference (sensory) to another (motor), 2) a transformation from one type of vectorial expression (covariant) to another (contravariant), 3) the above transformations in the case when the motor system is higher dimensional than the sensory?

The problem of sensorimotor coordination, rooted in the Cartesian school of thinking (see al) part of Fig. 1), is tensorially redefined, based on the following postulates: a) Vectorial expressions are assigned within the CNS to external objects, such as a limb-displacement (d–e2), which are inherently invariant to coordinate systems (cf. vectors in Fig. 1.; \( r_1, p^i, i_k, e^n \), where co- and contravariants are denoted by lower and upper index, respectively). b) The internal vectors differ. For instance, the reception vector \( r_1 \) (d–e4) is covariant-type (since its components are established by independent sensors) and is expressed in an i=2 dimensional frame, while the execution vector \( e^n \) (f4) is contravariant since its components must physically assemble the displacement by means of the three-segment limb (d–g2). This vector is expressed in an n=3 dimensional motor system.

Different vectorial expressions of the same invariant, by definition, are related to one another via mathematical devices called tensors. Thus, for instance, the transformation, as a whole, can be characterized by a sensorimotor tensor \( S^n \) that transforms \( r_1 \) to \( e^n \) (a–b2). Still, the question remains of how the CNS implements such tensor-transformations, enabling the sensorimotor system to arrive at a higher dimensional contravariant-type \( e^n \) vector, expressed in the motor system, starting from a different, possibly lower dimensional and covariant-type \( r_1 \), which was expressed in a sensory frame.

Tensor network theory suggests an answer by a three-step scheme. (1) A covariant-contravariant tensor transformation in the sensory frame, via a contravariant sensory metric tensor \( g^{ij} \), that transforms covariant sensory reception \( r_1 \) into contravariant perception \( p_i^j \) (c–e5). (2) A sensorimotor coordinate-system transformation via a covariant embedding tensor \( c_{Wk} \) (that can be applied regardless of the difference in dimensionality, or the direction of the axes, of the sensory and motor frames). The embedding tensor \( c_{Wk} \) transforms the covariant sensory perception into covariant motor intention \( i_k \) (a–b 5–10). (3) A covariant-contravariant transformation in the motor frame via a contravariant motor metric \( g_{ik} \) that transforms covariant intention into contravariant motor execution vector \( e^i_k \) (d8).

Tensor-transformations are represented in the diagram by five different expressions simultaneously. 1) By tensor equations, using coordinate-system aspecific notation (block d–g10). These apply to any and all sensorimotor systems no matter their reference frames. 2) Verbally, providing an intuitive heuristic description of the
function (reception, perception, intention, execution). 3) By Euclidean pictograms, visualizing the geometrical characteristics of different reference frames and vectorial versions (d–g 3–6), see also the covariant embedding block (a10). 4) By neuronal networks, demonstrating that abstract tensorial functions can be implemented by relatively simple neuronal networks. The contravariant sensory metric $g^i_j$ is placed into a tectal network (c5–6). The covariant sensorimotor embedding tensor $c_{ik}$ is shown to be implemented by the lower strata of a cortical network. The neurons of the upper strata (a5–8) and the ascending proprioception system do not directly take part in the transformation: they play a role in the process of building the necessary networks, to be discussed elsewhere. The generalized inverse of the covariant motor metric tensor $g_i^k$ is shown to be implemented by the cerebellar connectivity (d7–8).

5) By numerical vectors and matrices, in order to provide a scheme where a sensory input vector can be numerically traced into the motor output of the system. The initial $r_i$ vector-components are scaled into a range of one hundred to facilitate thinking of them as firing frequencies. While the scaling is arbitrary, the relationships of all vector-components do quantitatively correspond to the geometric features of depicted frames and co- or contravariant vectorial versions, cf. blocks (d–g 3–6) and (a10). Similarly, the matrix components of sensory metric $g^{ik}$ (c–d 5–6) and the synaptic coefficients of covariant embedding tensor $c_{ik}$ (a–c 5–8) are quantitatively determined by the sensory frame (d–c 2–3) and its geometric relationship to the motor frame, using a unit-vector $u$ in the latter (a10). Although the sensory two-space can be embedded into a motor three-space in an infinite number of ways, for the geometry of the motor hyperspace the generalized inverse of the covariant metric provides a unique solution (Ben-Israel and Greville, 1980) (d8). While the (1)–(3) matrices perform qualitatively distinct functions and are implemented by separate networks, mathematically they can be contracted into a single matrix $S^m$ acting as the total sensorimotor tensor (e10). Note that all matrices implementing a generally describable tensorial function, can be constructed by an infinite number of variations of the particular neuronal networks (cf. Fig. 2 in Pellionisz and Llinás, 1982a). The numerical vector-matrix example also serves as a demonstration that, based on the understanding gained from the abstract handling, an actually working software or hardware implementation of such tensorial blueprints is possible.

### 3.3 The Contribution of Tensor Network Theory to a Conceptually and Formally Unified View of the Brain

Below, a selection of the major contributions of the tensorial approach to neuroscience is given. Most of these features are demonstrated in Fig. 1. For further substantiation, however, one may wish to turn to the referenced worked-out examples.

**Tensor Approach Provides a Neuronal Network Theory.** A network theory is essential for neuroscience. It is hardly conceivable to understand brain function without being able to formally represent sets of neuronal connections, where the description of the particular system (e.g. a cerebellum of a frog) would simultaneously be conceptually tied to a general functional interpretation of all such networks (e.g. the cerebellum in any species).

Networks have long been described mathematically as matrices. They are now interpreted as not just matrices but tensors, i.e. transformation matrices which change one vector to another in relation to the invariant. For instance, matrix $L$ in (c–d 5–6) transforms the vectorial expressions shown in boxes (d–c 3–6). These changes in the physical relationship of vector-components to an invariant can now be made visually explicit in vectorial pictograms (d–g 3–6), see also Pellionisz and Llinás, (1982a,b). A tensor-transformation of a vector by a matrix may change a vector to another in several ways: a) Both the input and output vectors are expressed in the same coordinate system but by a different vectorial version. For example, the $g^i_j$ tensor expressed by matrix $L$ transforms a sensory reception vector $r_i$ to another, sensory perception vector $p^i_j$, but the former is covariant, while the latter is contravariant. Likewise, the $g_{ik}$ tensor, expressed by matrix $M$ transforms the $i_k$ motor intention vector into an $e^n$ motor execution vector, where $i_k$ is covariant and $e^n$ is contravariant.

b) The input- and output vectors are expressed by the same vectorial version but in a different coordinate system. c) The input-output vectors are expressed both in different frames and by a different version, as the tensor $c_{ik}$, implemented by matrix $M$, transforms contravariant sensory vector $p^i_k$ into covariant motor vector $i_k$.

The chief benefit of the tensor approach as a network theory is that a single tensor interprets a whole group of matrices as implementing the same general tensorial transformation. E.g. the two-by-two matrix $L$ applies to the sensory frames with a 120° axes, as in Fig. 1., while the general expression $g^i_j$ applies to frames with any angle. Thus, tensor approach is applicable to both a concrete individual network, described by a matrix, and a general network-type, described by a tensor: “The importance of tensors in mathematical physics and geometry rests on the fact that a tensor equation is true in all coordinate systems, if true in one” (Synge and Schild, 1949). In Fig. 1., for instance, the transformations are characterized both by the reference-frame-dependent 1–2–3 matrices and the general tensor formalism in the lower right bow (d–g 9–10).

Tensor theory of neuronal networks is conceptually comparable to Kron’s general treatment (1939) of electrical networks, cf. Pellionisz and Llinás, (1982b). His approach was that of a “theoretical engineer” who wanted to solve a whole group of problems in a general manner. This unusual ambition led him to complex general treatments of problems that could be solved in a much simpler way on a case-by-case basis. Since engineering has the trend of moving from the abstract understanding to the concrete task, Kron’s efforts represented a “reverse engineering approach”. As a result, his work was neither very highly valued nor even sufficiently understood. In neuroscience, however, we are assigned precisely that “reverse engineering” task; starting with a particular brain, we must move into the direction of the abstract, towards the brain. Thus, Kron’s legacy is of great usefulness.

**Tensor Approach Introduces the Use of Covariant (Sensory) Analysis and Contravariant (Motor) Synthesis.** The mathematical theory of covariant and contravariant vectors is not new. These terms were originally used by Sylvester in the 19th Century, (Wrede, 1972), and the refined concepts were utilized in relativity (Einstein,
However, covariant and contravariant vectorial expressions in neurobiology were distinguished by this author only in early 1979, after initiation of the tensor network theory of the brain (Pellionisz and Llinàs, 1978, 1979a). The two types of vectors were put into use (Pellionisz and Llinàs, 1979b, 1980a,b), suggesting that the covariant analysis and contravariant synthesis was a general principle of organization of the CNS (Pellionisz and Llinàs, 1980a).

Tensor Approach Provides a Principle of the Overcomplete Organization of the Brain. The decomposition of an invariant into independently established covariant components allows the procedure of covariant embedding (Pellionisz and Llinàs, 1980a) of an embedded space into a possibly much higher dimensional embedding space. This procedure is based on the fact that in the case of an m-dimensional object n coordinate axes may be used to arrive at the covariant components, where n may be any number. The original example was provided in Fig. 4 in (Pellionisz and Llinàs, 1980a) for embedding a two-dimensional invariant, a location in a plane, into a three-dimensional overcomplete space. A similar example is provided in Fig. 1, where the two-dimensional displacement of a 3-segment motor limb requires the expression of the invariant in the sensory two-space (by r_i and p_i), in the motor three-space (by i_k and e^s_k) and the embedding of the sensory space into the motor space. The latter is accomplished by the sensorimotor tensor c^k_k implemented by matrix 2.

Tensor Approach Geometrically Redefines Coordination as Covariant Embedding and Covariant-Contravariant Metric Transformation. A coordinated act could be defined as a goal-oriented action implemented with more than the necessary number of executors — e.g. an action where an externally presented four-dimensional (space-time) target-point is spatiotemporally matched by a displacement of a part of the body. Such an act requires the brain to command a very high dimensional musculoskeletal system, easily involving as many as 10^5-10^6 motor effectors.

While a whole volume may be devoted to the subject of “motor coordination”, without mathematically defining it (Llinàs and Simpson, 1981), the tensor paradigm provides such definition and an explanation of its implementation by a neuronal mechanism (Pellionisz and Llinàs, 1980a,b, 1982a). Coordination can be defined as a covariant embedding (that expresses the movement intention in a higher dimensional motor hyperspace than the sensory system), followed by a covariant-contravariant transformation that expresses the movement in executable contravariant form. In Fig. 1. these transformations are implemented by c^k_k embedding-tensor and g^k^m motor metric transformation.

Tensor Approach Offers a New Concept for the Function of the Cerebellum: It Acts as a Contravariant Metric Tensor. This approach provided a concise yet expandable theory for the function of the cerebellum (Pellionisz and Llinàs, 1980a,b, 1982a). It was suggested that the covariant motor intention vectors are transformed into contravariant motor execution vectors by the neuronal network of the cerebellum, which acts as a contravariant metric tensor of the brain motor hyperspace. In Fig. 1, such motor-metric transformation is described by g^m^k tensor, expressed as matrix 3, implemented by the neuronal connectivity of the cerebellum. The tensorial explanation of cerebellar function is to be compared to a limited number of theories capable of providing both a mathematically definable and neurobiologically feasible model for the function of this organ (for reviews, see Llinàs and Simpson, 1981, Llinàs, 1981, Pellionisz, 1984c).

Tensor Approach Provides a New Axiom and Formal Treatment for Space-Time Representation in the Brain. The tensor concept and formalism is eminently capable of describing space-time objects. As in physics (Weyl, 1952, Schrödinger, 1950), coincidences of moving targets with body movements can also be represented tensorially in neuroscience (Pellionisz and Llinàs, 1982a). It is important to note that the purpose of the treatment there was not the external description of space-time events, but the explanation of the inner workings of the mechanism in the terms natural to the neuronal system which implements the function. An alternative was provided to the general assumption that in cells functioning the CNS utilizes coordinate systems similar to those in classical mechanics. The former assumption led to suggestions that the cerebellum acts as a clock (Braitenberg, 1961). In a re-assessment of this pioneering concept, it was proposed that the brain does not separate space- and time-coordinates, but assigns space-time coordinates to event-points (Pellionisz and Llinàs, 1982a,b). In order to transform such asynchronous coordinates (e.g. from covariant vectors into contravariant motor expressions), the brain must be furnished with neural networks, such as the cerebellum, acting as contravariant motor space-time metric tensors (Pellionisz and Llinàs, 1982a).

Tensor Approach Proposes a New Concept for the Function of the Superior Colliculus. The Optic Tectum as a Contravariant Sensory Space-Time Metric. A new framework could be provided for the functioning of preprocessors in the sensory part of the brain, cf. neuronal circuitry that implements matrix 1 (c-e 3-6). It was argued (Pellionisz and Llinàs, 1982a) that covariant space-time sensory receptor information has to be transformed into contravariant expression. Such transformation is necessary in order to yield quantities which express the external invariant displacement d in the form of d=r_i p_i, the inner product of the co- and contravariant vectorial versions. It has been suggested (Pellionisz and Llinàs, 1982a,b) that the neuronal networks of the superior colliculus (optic tectum) are capable of performing such a contravariant metric transformation; thus, this organ may act as a sensory space-time metric tensor. The suggestion has been elaborated in Pellionisz (1983).

Tensor Approach Explains the Functional Organization of Sensorimotor Systems with Different Sensory and Motor Frames. In some sensorimotor systems it is possible to physically demonstrate that the sensory frame of reference is different from the motor system of coordinates, both in the number of coordinate axes and in their directions. Such is the vestibulo-ocular system where the six semicircular canals constitute a generally non-orthogonal set (Curthoys et al., 1975). It is a physical fact that each canal is sensitive to the orthogonal projection (covariant component) of the head acceleration into its plane. In turn, the motor system of the two eyes is at least a twelve dimensional system, working with physically executable contravariants, and the extraocular muscles form an even more obviously non-orthogonal arrangement (Ostriker et al., 1982). When considering also the neck muscles, the gaze-stabilizing system (including VOR) is the epitome of a covariant-contravariant tensorial scheme
with different non-orthogonal sensory and motor systems (Pellionisz and Llinás, 1982a, Pellionisz, 1984b).

The tensorial approach has been offered to VOR research (Pellionisz and Llinás, 1979a, 1980a) since this is a system which clearly calls for a general description, free of coordinate system-specific notation. In a simplifying effort toward adopting tensorial treatment of parallel systems, orthogonal frames of reference were applied to this system, not considering covariant and contravariant vectorial versions (Schultheis and Robinson, 1981). A further step is the extensive use of particular matrices, while largely refraining from the abstraction of tensors (Robinson, 1982). Specifically for the extraocular muscle system a computer model was developed, applicable to all species since it uses a notation that is free of specific frames of reference (Ostricker et al., 1982, Pellionisz, 1984b).

**Tensor Approach Abstracts Descriptive Neurophysiological Terms: Reception, Perception, Proprioception, Intention, and Execution into Vectorial Expressions.** The neuronal message at different points of a sensorimotor system can be mathematically stated in terms of vectors (Pellionisz and Llinás, 1982a). Such vectorial information, while mathematically abstract, can also be expressed verbally in order to convey a heuristic intuitive meaning. For instance, in a system where the sensor- and motor activities are expressed in different frames of reference, a single external invariant, such as a location in physical space is carried by four different vectors: covariant sensory (reception), contravariant sensory (perception), covariant motor (intention) and contravariant motor (execution), as seen in Fig. 1.

**Tensor Approach Offers Implementable Blueprints of Neuronal Systems.** While a tensor-transformation representation of a neurobiological function is given in a non-coordinate-specific manner, the transformation can be numerically expressed in any particular frame of reference. For instance, $g^{i j}, e_{j k}, e^{k i}$ tensors in Fig. 1 can be "spelled out" in matrices 1–2–3. Such arrays of quantities, which define physical connections between input and output elements, can also be implemented by either software computer models or hardware especially created for this purpose. Such implementations of tensorial neurobiological schemes will lead to new types of robots: indeed, brain-like machines.

### 3.4 Perspective of the Tensor Approach

In an attempt to provide a rounded presentation of the contribution of the tensor approach to the task of connecting neuroscience with robotics, it may also be important to indicate its potential for future growth. Below such a brief outlook is given, containing also some conceptual considerations which could not yet have been mathematically elaborated.

**Genesis and Modification of Metric Tensor Networks.** The earlier expressed view (Pellionisz and Llinás, 1979) is emphasized that an understanding of brain function may be dependent upon the resolution of the geometrical properties of its hyperspace. Several key features of brain function (the transformation of covariants into contravariants, etc.) are intimately related to the networks acting as the metric tensor of the functional hyperspace of the brain (cf. network-matrices 1 and 3 in Fig. 1). Indeed, it is known from tensor analysis that the metric tensor is the main mathematical device that determines the properties of a space (Levi-Civita, 1926, Wrede, 1972). Thus, the function of different neuronal organizations could already be corroborated with such metric tensor transformations, cf. models for the function of the cerebellum (Pellionisz and Llinás, 1982a or tectum (Pellionisz, 1983).

A fundamental problem, however, is how neuronal networks, acting as metric tensors (as in Fig. 1.) arise in the brain. As noted (Pellionisz and Llinás, 1981), the cerebellum is an excellent candidate in which to search for an answer because the musculoskeletal geometry is innate and thus the cerebellar metric may be ontogenetically constructed. Indeed, a need for functional modification of an existing metric is obvious in the case of the cerebellum, cf. Llinás and Pellionisz, 1984. Given that the motor frame is dependent on the position of the limb, the motor hyperspace is curved. This means, in practical terms, that the metric cannot be constant; the connectivity matrix must be dependent on the position of the motor vector in this hyperspace. The climbing fiber system has been considered as actively changing the curvature of the hyperspace by altering the physiological transformation of motor vectors through the cerebellar metric (Pellionisz and Llinás, 1980b, Pellionisz, 1984b).

It was noted that the key to the building of metrics lies in distinguishing between covariant and contravariant expressions (Pellionisz and Llinás, 1981). The theoretical work to develop the necessary formalism has since progressed toward completion and the paradigm, yielding the methods of constructing metrics and transformation matrices (e.g. 1–2–3 in Fig. 1) will be the subject of a forthcoming publication of the series of Pellionisz and Llinás (1979, 1980a, 1982a).

**Motor Execution and Motor Proprioception Vectors: Incorporating Covariant Metrics.** It was proposed that motor systems execute contravariants while sensory systems report covariantly on the emerging invariant (Pellionisz and Llinás, 1980a,b). This implies that the functioning of a motor system which is endowed with covariant proprioception (using the same frame of reference as the activators) expresses, in fact, a covariant metric since to any contravariant motor execution vector its covariant proprioceptive counterpart is produced. However, such metric is insufficient, in itself, for organizing the motor action since 1) it exists only implicitly in the functioning, 2) it is the wrong version of the metric, as opposed to the contravariant metric necessary for cerebellar-type coordination.

The Refraction-angle and Impact of a Vector on a Metric. It follows from the different properties of the two vectorial versions assigned to the same invariant that they are not interchangeable in expressing it. For example, if a given covariant vector is used as if it were a contravariant, it will assemble a different invariant than the one to which it was originally assigned. This may be the case with cerebellar lesions (Pellionisz and Llinás, 1980a), when in the absence of a cerebellar metric (dysmetria) the covariant motor intention vectors are executed as if they were contravariants. This results in movements in a wrong direction and with a wrong amplitude. If the angle between a physically executed covariant and contravariant is called here "refraction angle" of the vector $v$ on the metric $g^{i j}$, and denoted by $\alpha$ (where both $v^i$ and $v^j$ are assigned to the same invariant, thus $v^i = g^{i j} v^j$), then $\alpha = \cos(\gamma)$ is defined as the "impact" of the vector $v$ on $g^{i j}$. 
The refraction angle is not constant but depends on the intended direction (Pellionisz and Llinás, 1980a,b). For directions that are the so-called eigenvectors of the metric tensor, the deviation should be zero. This derives from the conceptual mathematical definition that an eigenvector of a matrix is the input vector for which the output is of the same direction (only the amplitude may be different). Therefore, if the refraction can be established, it may serve as a measure of how "close" a vector is to an eigenvector. Such a measure is of supreme theoretical significance, since the eigenvectors are most important in characterizing a system (Pettofrezzo, 1966). The importance of eigenvectors has been greatly exploited in brain theory (Anderson et al., 1977; Anderson and Mozer, 1981).

Since in an orthogonal frame of reference the metric tensor is the Kronecker delta, in such frames every direction is an eigendirection since the co- and contravariant vectors are identical. Thus, if the refraction angle is measured in different directions, the non-orthogonality of a system can be characterized without having to identify the coordinate-axes: a) If the impact is total into all directions, the system is orthogonal. b) If the impact is less than total in any direction, the system is non-orthogonal. c) An eigenvector is found wherever the impact is total. The refractation and impact is being used to quantitatively reveal the non-orthogonal character of the oculomotor coordinate system (Ostrik et al., 1982).

**Tensor Approach and the General Conceptualization of Brain Function.** The central question that cannot be as extensively treated here as it deserves, yet too important not to be put, at least, into perspective, is "How can the tensorial approach help crystallize concepts about the functioning of the brain into mathematical formalism and implementable schemes?" The essence of the tensor concept is that the brain is being looked at from the viewpoint of geometry. As pointed out earlier (Pellionisz and Llinás, 1982b), comparable ideas have already been mentioned — i.e. for the cerebellum (Greene, 1972; Llinás, 1974). To match the general geometrical concept with a suitable formalism, tensor analysis was suggested (Pellionisz and Llinás, 1979). As a matter of course, other methods can also be applied to related geometries. Therefore, to avoid giving an unfair impression, a comparison appears unavoidable even if it has to be very concisely given within the scope of this paper.

**Comparison of Tensor Approach with other Geometric Theories: Representation, Modeling, Mapping, Differential Geometry, Lie Algebra.** Generally, the most significant difference in the above alternative approaches may be that they do not offer as inherent a network theory as the tensor approach, either at the numerically concrete or reference-frame-invariant abstract levels of describing neuronal circuits. Another difficulty lies in distinguishing between the sensory and motor neuronal signals, treated in tensor theory as covariant and contravariant versions of vectors. In general, while some of the alternative approaches are either conceptually or formally rather attractive, one that is simultaneously adequate from both points of view is rarely found. Formally, some of the above approaches are mathematically more diffuse than tensor analysis (e.g. modeling). At the other extreme, some mathematical structures may prove to be too abstract for practical applications in biology, or even in engineering. Such approaches are usually less potent in heuristic power and intuitive meaning than tensor theory (Lie algebra or differential geometry; cf. Lefschetz, 1977; Spivak, 1979). Finally, few approaches are as success as tensor analysis in providing conceptual-formal means for the abstract description of natural phenomena in a manner that is accessible to engineering.

Specifically, representation theory is an eminently appropriate encompassing concept. However, its mathematical generality makes it difficult to use for a quantitative network theory of particular networks.

It has often been said (especially in connection with the "learning" theories) that the brain is a "model-reference adaptive control system" (Ito, 1970). This is the basis of the "internal model" paradigm. While modeling appears to be a conceptually attractive approach, it is not as homogeneous in its methods as tensor theory and consequently does not naturally yield a principal formalism matched with the idea. At the same time it conveys a one-to-one type of representation that is conceptually questionable for the brain.

Several workers use the terminology of topological mapping (Cooper, 1974, von Malsburg and Willshaw, 1976, Schwartz, 1977, Amari, 1980, Cowan, 1974), usually meant as a one-to-one representation from one domain to another, so that certain topological features are common (Lefschetz, 1977, Spivak, 1979). While such relationships can be described, for example, by differential geometry and even tensor analysis (Coburn, 1970), this type of one-to-one, invariant-invariant relationship may not always be the issue in the CNS. For instance, in the case of "somatotopy" (e.g. of the cerebellar cortex Snider, 1952) it is difficult to show a one-to-one relationship of points in topologically contiguous surfaces of a cortex with points of the external space or the segments of the body: a one-to-one representation may be as frequent as a many-to-one, one-to-many, or, indeed, a many-to-many type of projection.

The tensor concept appears to be more flexible than only being capable of characterizing an invariant-invariant type of relationship. Vectors, as tensors of rank one, express an invariant in a vectorial manner. This is a one-to-many type relationship, such as a sensory covariant expression where one physical point is expressed by many components. In turn, a tensor may express a many-to-one relationship, e.g. in the case of motor contravariants where many motor components assemble one displacement point. In addition, however, tensorial vector-vector relationships (the majority of neuronal expressions between the sensory input and motor output) represent many-to-many type relationships. Thus, the tensor approach eliminates the need to search for one-to-one type mappings — a frequently expressed frustration (Lashley, 1951, Moore, 1980).

**General Conceptualization: the Brain as a Device Implementing Metageometries.** In the first section of this paper, major examples of the presently available choice of general concepts in brain theory were overviewed. To complete the cycle, a summarized assessment of how the tensor approach fares in this respect may be warranted here.

The tensor approach attempts to lead (with enough time and effort invested in it) to a synthesis of the most significant general features of brain function, painstakingly
worked out in theories of control, communication, information, learning and self-organization. Moreover, it wishes to do it with the use of a generalized version of the most used and most potent vector-formalism.

It appears, that the concept which is general enough to encompass the above principal features, yet expressible both generally and in numerical details, is the one stating that in the brain geometries organize one another. Elaboration of such processes and procedures (called organization of Metageometries in a forthcoming paper of the series (Pellionisz and Llinás, 1979, 1980a, 1982a)) will necessitate much further effort. Some indications of the potential inherent in geometric concepts coupled with tensor formalism, however, can be concisely given at this point.

That biological systems are governed by an underlying geometry and that one geometry is capable of organizing another, is proven now by the double helix. The molecular geometry of the helices organizes the physical geometry of not just the body in general, but also the structure of neuronal networks in particular. Networks, in turn, generate another, an internal functional geometry. This, ideally, must match the system of relationships existing in the external world. When the match is perfected by the brain, the least possible friction will exist between this functional “hypersurface” and the “hypersurface” of the relationships found in the external reality. The goal of brains may thus be seen as the optimization, for the individual, the goal of evolution: “the survival of the fittest metric”.

Tensor Analysis: Formal Treatment of Geometries Organizing one Another. Related geometries do not have to be expressed in the same medium; they may not be on the same scale, may not have the same number of dimensions and may not be isomorphic. The establishment of relationships between geometries (i.e. physical geometries of invariants and abstract geometries in hyperspaces) requires, however, a suitable medium through which both can be treated. Tensors, e.g. expressed by neuronal networks, are suitable mathematical devices for the purpose, since they relate invariants to vectors, vectors to vectors, vectors to invariants and thus invariants to invariants. The formal whirls of the approach are best contained in the mathematical device of the metric tensor: since a) features of the geometrical properties of the space can be deduced from this concise formula (Einstein, 1916, Levi-Civita, 1926, Coburn, 1970), b) the matrix of the metric tensor can be implemented by neuronal networks.

Contribution of the Tensor Theory of Organization of Hierarchic Geometries to the Cohesion of Principal Concepts in Brain Theory. While an approach that re-states a concept may be considered superfluous, one that is capable of re-stating several disparate concepts in a homogeneous unifying manner, is usually welcome in science. Putting tensor theory of organizing hierarchic geometries to this test below, it is found not incompatible with existing principal concepts.

Self-Organization can be seen geometrically as the construction of a metric tensor within the system. Other than considering an organized system as having a defined functional geometry (embodied in a metric), this interpretation differs from theories of self-organization mainly in the aspect that it de-emphasizes the autonomous features of the emergence of such metrics. Indeed, even the innate structuro-

functional properties of the CNS appear to be organized by genetic information which is provided by the geometries inherent in molecular biological mechanism. As acquired structuro-functional properties, theories of adaptive systems emphasize the organization of the CNS by external geometries. On the other hand, control theories put emphasis on the ability of the CNS to project its internal functional organization onto external systems (e.g. physical movements). These distinctions among self-organization, adaptation and control theories may appear avoidable if their common principle is put into the limelight: that in each case geometries interact with one another in order to become homeomorphic if not isomorphic. It will be pointed out elsewhere how metric tensors, embodied in neuronal networks, may interact through invariants or through common vectorial expressions in a manner that they organize, generate and modify one another. This will show that such “metagenesis” of geometries is applicable not only for generating a neuronal functional geometry that matches a physical one. It also seems to apply to the case when a neuronal metric tensor (e.g. a contravariant one) can be both duplicated and complemented (yielding both a contravariant and a covariant metric) by the same network-procedure. Such processes may be seen as the basis for generating a hierarchy of matching geometries in the CNS that interact with one another, as well as with external geometries.

Intelligence may thus be considered in the above geometrical terms as the ability to implement hierarchic metageometries in brain hyperspaces that are ill-organized amorphous and dysmetric in their pristine states. The intelligence of a system thus may be measured by a) how many hierarchic hyperspaces it can organize, b) how precisely these geometries match.

Communication and Neuronal Information are also compatible with a geometrical interpretation. Neuronal messages are treated as vectorial representations of invariants. The activity of a single neuron, therefore, is considered not as the message itself but only its vectorial component. Neuronal communication, thus, may be defined as the exchange of messages through vectorial channels. As for Neuronal Information, it is often stated that the brain is an information processing device. Yet information is defined in probability theory as the negative binary logarithm of the probability of an event (Shannon, 1948); thus, the biological relevance of this term may be questioned (Moore, 1980). From a geometrical point of view, the information content of an incoming neuronal message is its geometrically formative part on a given metric — the part which passes the “novelty filter”, defined in the field of artificial intelligence (Kohonen et al., 1981). The geometrically defined information may thus be compared with the impact (a measure of the importance) of the vectorial message. They are comparable since neither depends only on the message itself, but on its relationship to the geometry toward which it is applied.

In turn, learning may then be defined as the process of internalization of neuronal information. As geometrical definition implies that when the internal and external geometries match, further learning is both unnecessary and impossible; no message, however important it may be, can present formative information to a perfect geometry. The formative effect of a vectorial message on a non-matching meta-geometry thus will depend both on the information it represents to the geometry that
is being organized and on the impact (importance) of the message, as measured by the organizing geometry.

Memory, Associative Storage and Recall are terms exactly defined in the field of artificial intelligence (Kohonen, 1967), and need not be belabored in the scope of this paper. Nevertheless, since vector-matrix expressions generally use orthogonal frames of reference in the extensive literature of this field (Kohonen, 1967, Anderson et al., 1972, Willshaw, 1981, Kohonen et al., 1981), it seems plausible to tensorially reinterpret the available results, using non-orthogonal general coordinates, in a forthcoming publication.

Implementation of Devices which Generate Metageometries. Both tensor theory of the brain in general and this paper in particular are directed, at the present time, toward gaining a directly utilizable understanding of some specialized subsystems of the brain such as the cerebellum (Pellionisz, 1984a,c). It will require an extensive effort to actually implement such understanding even at the prototype level. Nevertheless, it seems already likely that brain-like machines will, in time, be capable of such functions that are all too familiar to our minds but that we never expected from machines.

References


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Ecphes, J. C. (1981). The modular operation of the cerebellar neocortex considered as the material basis of mental events. Neuroscience 6, 1839-1856.
ANDRÁS J. PELLIONISZ


**Brain theory and robotics**


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