(39) Pellionisz, A. and Llinas, R. (1979) A Note on a General Approach to the Problem of Distributed Brain Function. Matrix and Tensor Quarterly, The Journal of the Tensor and Matrix Society of Great Britain. 30, 48-50.

A NOTE ON A GENERAL APPROACH TO THE PROBLEM OF DISTRIBUTED BRAIN FUNCTION

by

A. Pellionisz * and R. Llinás *

A tensor network approach is adopted to convey the fundamental principles and properties of neuronal systems.

INTRODUCTION

A fundamental task of a theory for the central nervous system (CNS) is to provide an organizational principle for the distributed and parallel character of the brain. Neuronal networks are commonly characterized as a system of serially connected elements interacting in a many-to-many fashion[7]. However, the CNS, and most conspicuously its laminar cortices, are *not* serially organized but, rather, operate in a *parallel* manner[4] such that the arrays of *distributed elements* may generate specific functions as *emergent* properties of their connectivity[3].

Technically, arrays of neurons can be handled, formally, in several convenient ways; e.g., *patterns* of activities can be described over large matrices of neurons by either layer-by-layer computer simulation analysis[5] or by theoretical treatment[1].

From a mathematical point of view, arrays of neurons can be readily approached by ordered sets of different ranks. For example, a set of I number of neurons may be represented by an array of rank one; $\overline{M}_i$ where $i \in (1, I)$ and the symbol M is given a single bar to show the rank of the array. Here the kth single component of the column-vector $\overline{M}_i$ is denoted by $\overline{M}_{k\bullet}$. This value represents a scalar measure of the level of activation of the kth neuron. (The activation $\overline{M}_{k\bullet}$ may be measured in spiking neuronal systems by the spike discharge *frequency*, and in non-spiking subsystems by the value of the graded membrane potential.) Similarly, an array of rank two (a conventional matrix) may be used to describe the system of connections between two sets of neurons: If J number of P neurons are connected to I number of M neurons, then the connection between the ith M neuron and the jth P neuron may be denoted by $\overline{\overline{\Psi}}{}^{i\bullet}_{j\bullet}$; thus the matrix $\overline{\overline{\Psi}}{}^{i}_{j}$ describes the total set of connections. Assuming linear relations, operations on $\overline{M}$, $\overline{P}$ and $\overline{\overline{\Psi}}$ can be handled by multilinear algebra. However, a fundamental question remains as to what mathematical entities neuronal ensembles represent which are being handled by linear algebraic methods.

BRAINS AS TENSORIAL SYSTEMS

The *existence* of a single underlying entity capable of representing any set of particular neuronal networks is implicitly assumed in neuroscience; i.e., data derived from *particular* neuronal networks are generalized to the *set* of neuronal networks

* Department of Physiology and Biophysics, New York University Medical Center, New York, NY 10016, USA.

(from *a* brain to *the* brain). This to us is equivalent to regarding the brain as a *geometric object* in the Kronian sense [2] where every individual connectivity-matrix is considered a particular expression of the same reference-frame invariant vector-relationship: the network *tensor*.

Thus, our central tenet is that the brain is a tensorial system, using vectorial language [6]. A simple reason for this assumption is as follows. Considering the system of connections among two sets of neurons as identified by a matrix, this, in some cases, may consist of as many as 10^{15} elements. While we *know* that a particular matrix *is* actually incorporated in each brain, it is obvious that such an astronomical number of matrix elements *cannot* be pre-specified biologically. Hence the question is, how do matrices of a given class develop such that the amazingly *invariant vector-vector function* (in general terms, the environment-behaviour response) arises without actually specifying the values of each matrix element? We assume that the neuronal networks are endowed with tensorial properties. Thus, since *all* individual matrices of a given class incorporate a tensor, the particular connectivity matrices (which are clearly not identical) are **expressions in different frames of reference of the same tensor.** The frames of reference in the multi-dimensional input and output spaces are given by the individual neurons. From it, it follows that both the input and output of the nerve net are represented as vectors (curves) in the multi-dimensional input and output frequency-spaces (neuro-hyperspaces). The relation of these vectors is identified, in the particular selection of frame of reference, by the connectivity matrix of the input and output neurons.

Thus, in the above-mentioned example of M and P neurons, the connectivity matrix $\overline{\overline{\Psi}}$ determines the vector-vector relationship

$$\overline{P}_j = M_i \, . \, \overline{\overline{\Psi}}{}^i_j$$

where we can identify $\overline{\overline{\Psi}}{}^i_j$ as the **network tensor.** (In the above equation the Einstein summation convention is used.) In any particular case of the class of network, Ψ is given as an array of rank two (connectivity matrix). However, since the vector-vector relationship is a biologically meaningful entity and is represented by a tangible neuronal circuitry (which is a geometrical object), it has to be, and indeed it *is*, remarkably invariant to the particularities of the individual matrix. Thus, all $\overline{\overline{\Psi}}$ matrices are incorporations of a tensor. Since tensors are reference-frame invariant vectorial relations, it *follows* from the fundamental principle of invariance that their expressions in different matrices must have certain transformation properties (that guarantee the invariance of vectorial relations). It is emphasized, that since there is no way of knowing the transformation properties of one brain network to another we base our tensorial assumption not on the consequential transformation properties, but on the more fundamental **principle of invariance.**

In tensorial terms not only general *properties* of neuronal systems can be defined (reflexes, receptive fields, convergences, divergences) but also the system itself can be identified and, therefore, investigated from appropriate points of view. For example, a part of the brain known as the cerebellum can be investigated in the frequency hyperspace in which the matrix of cerebellar tensor specifies a curved set of trajectories. Cerebellar coordination of ballistic movements can be described as guiding the movement on to 'wired-in' trajectories of the vector-field, by virtue of the inhibition- and coordination-vectors, provided by the cerebellum[6].

Beyond appreciating the power that tensorial methods may provide in grasping the fundamental principles and properties of neuronal systems, it is aesthetically pleasing that tensorial laws, which yield the abstract language for describing the universe, should also be applicable to understanding our brain.

REFERENCES

1. *A. K. Katchalsky, V. Rowland and R. Blumenthal* Dynamic Patterns of Brain Cell Assemblies, Neurosci. Res. Prog. Bull., Vol. 12, pp. 1–187, 1974.
2. *G. Kron* Tensor Analysis of Networks, John Wiley, New York, 1939.
3. *R. Llinás* Eighteenth Bowditch Lecture: Motor Aspects of Cerebellar Control, The Physiologist, Vol. 17, pp. 19–46, 1974.
4. *J. von Neumann* The Computer and the Brain, Yale University Press, New Haven, Connecticut, USA, 1959.
5. *A. Pellionisz* Computer Simulation of the Pattern Transfer of Large Cerebellar Neuronal Fields, Acta Biochim. Biophys. Acad. Sci. Hung., Vol. 5, pp. 71–79, 1970.
6. *A. Pellionisz and R. Llinás* Brain Modeling by Tensor Network Theory and Computer Simulation. The Cerebellum: Distributed Processor for Predictive Coordination, Neuroscience (in press), 1979.
7. *S. Ramón y Cajal* Histologie du Systeme Nerveux de l'homme et des Vertebres, Vols. 1 and 2, 1911, Maloine, Paris, France.

* * * * *   * * * * *   * * * * *